

ON THE T -LEAVES OF SOME POISSON STRUCTURES RELATED TO PRODUCTS OF FLAG VARIETIES

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ABSTRACT. For a connected abelian Lie group \mathbb{T} acting on a Poisson manifold (Y, π) by Poisson isomorphisms, the \mathbb{T} -leaves of π in Y are, by definition, the orbits of the symplectic leaves of π under \mathbb{T} , and the leaf stabilizer of a \mathbb{T} -leaf is the subspace of the Lie algebra of \mathbb{T} that is everywhere tangent to all the symplectic leaves in the \mathbb{T} -leaf. In this paper, we first develop a general theory on \mathbb{T} -leaves and leaf stabilizers for a class of Poisson structures defined by Lie bialgebra actions and quasitriangular r -matrices. We then apply the general theory to four series of holomorphic Poisson structures on products of flag varieties and related spaces of a complex semi-simple Lie group G . We describe their T -leaf decompositions, where T is a maximal torus of G , in terms of *(open) extended Richardson varieties* and *extended double Bruhat cells associated to conjugacy classes* of G , and we compute their leaf stabilizers and the dimension of the symplectic leaves in each T -leaf.

1. INTRODUCTION AND STATEMENTS OF RESULTS

1.1. Introduction. A holomorphic Poisson structure on a complex manifold Y is a holomorphic bi-vector field π on Y such that the bracket $\{\phi, \psi\} = \pi(d\phi \wedge d\psi)$ on the sheaf of holomorphic functions satisfies the Jacobi identity. Holomorphic Poisson structures form an important class of Poisson structures, and they have recently also been studied in the context of generalized complex geometry and deformation theory (see [22, 28] and references therein).

A triple (Y, π, λ) , where (Y, π) is a complex Poisson manifold and λ a holomorphic action on Y by a connected abelian complex Lie group \mathbb{T} preserving π , is called a *complex \mathbb{T} -Poisson manifold*. A complex \mathbb{T} -Poisson manifold (Y, π, λ) gives rise to a decomposition of Y into the \mathbb{T} -orbits of symplectic leaves of π , also called *\mathbb{T} -leaves*, which are of the form $\cup_{t \in \mathbb{T}} t\Sigma$, where Σ is a symplectic leaf of π in Y (see §2.2 for the precise definition). While a complex manifold can not support non-symplectic holomorphic Poisson structures with finitely many symplectic leaves, as the degeneracy locus of such a Poisson structure, being a non-empty divisor, can not be the union of finitely many symplectic leaves, it is easy to construct examples of \mathbb{T} -Poisson manifolds with finitely many \mathbb{T} -leaves: for a complex torus \mathbb{T} , any smooth toric \mathbb{T} -variety with the zero Poisson structure is such an example.

If L is a \mathbb{T} -leaf in a \mathbb{T} -Poisson manifold (Y, π, λ) , the subspace \mathfrak{t}_L of the Lie algebra \mathfrak{t} of \mathbb{T} which is tangent to every symplectic leaf in L under the action λ is called the *leaf stabilizer* of $(\lambda \text{ in } L)$ (see §2.2). Every \mathbb{T} -leaf L in (Y, π, λ) admits a nowhere vanishing anti-canonical section, called the *Poisson \mathbb{T} -Pfaffian*, constructed as an exterior product of the Poisson bi-vector field π and vector fields on Y coming from any complement of \mathfrak{t}_L in \mathfrak{t} (see Remark 2.7), and the co-rank of π in L is equal to the codimension of \mathfrak{t}_L in \mathfrak{t} . For a given \mathbb{T} -Poisson manifold (Y, π, λ) , it is a natural and interesting problem to determine the \mathbb{T} -leaf decomposition of π in Y and the leaf stabilizers for the \mathbb{T} -leaves.

Let G be a connected complex semi-simple Lie group with Lie algebra \mathfrak{g} , and fix a pair (B, B_-) of opposite Borel subgroups of G and a symmetric non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on \mathfrak{g} . The choice of $(B, B_-, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ gives rise to a *standard* multiplicative holomorphic Poisson structure π_{st} on G , and the pair (G, π_{st}) is known as a *standard complex semi-simple Poisson Lie group* (see §6.1 for detail). The Poisson structure π_{st} is invariant under the action by the maximal torus $T = B \cap B_-$

by left translation. It is well-known [23, 25] that the T -leaves of π_{st} in G are the double Bruhat cells $G^{u,v} = BuB \cap B_-vB_-$, where $u, v \in W$, the Weyl group of (G, T) . Double Bruhat cells have been studied intensively and have served as motivating examples of the theories of total positivity and cluster algebras (see [3, 18, 21] and references therein).

The Poisson structure π_{st} on G projects to a well-defined Poisson structure, denoted by $\pi_{G/B}$, on the flag variety G/B of G . The T -leaves of $\pi_{G/B}$ in G/B have been shown in [20] to be precisely the open Richardson varieties, i.e, non-empty intersections $(BuB/B) \cap (B_-wB/B)$, where $u, w \in W$ and $w \leq u$ in the Bruhat order on W . Open Richardson varieties and their Zariski closures in G/B , called Richardson varieties, have also been studied intensively from the points of view of geometric representation theory, combinatorics, and cluster algebras (see, for example, [4, 5, 29, 31, 32, 33]). There are many other natural examples of T -Poisson manifolds associated to the Poisson Lie group (G, π_{st}) , including the generalized flag varieties G/P [20], where P is a parabolic subgroup of G , twisted conjugacy classes in G [16, 35, 36], symmetric spaces of G [15], the wonderful compactification of G when G is of adjoint type [15], and the variety of Lagrangian subalgebras [15, 39]. In these examples, the T -leaves, and leaf stabilizers in some cases, have been determined by somewhat ad-hoc methods (but see [52] for the method of *weak splittings* in the study of T -leaves and symplectic leaves for a class of Poisson structures including $\pi_{G/B}$ on G/B). .

In this paper, we describe the T -leaves and the leaf stabilizers for four series of T -Poisson manifolds associated to a standard complex semi-simple Poisson Lie group (G, π_{st}) , respectively denoted as

$$(1.1) \quad (F_n, \pi_n), \quad (\mathbb{F}_n, \Pi_n), \quad (\tilde{F}_n, \tilde{\pi}_n), \quad (\tilde{\mathbb{F}}_n, \tilde{\Pi}_n), \quad n \geq 1.$$

When $n = 1$, we have

$$\begin{aligned} (F_1, \pi_1) &= (G/B, \pi_{G/B}), & (\mathbb{F}_1, \Pi_1) &= ((G \times G)/(B \times B_-), \Pi_{(G \times G)/(B \times B_-)}), \\ (\tilde{F}_1, \tilde{\pi}_1) &= (G, \pi_{\text{st}}), & (\tilde{\mathbb{F}}_1, \tilde{\Pi}_1) &= (G \times G, \Pi_{\text{st}}), \end{aligned}$$

where $(G \times G, \Pi_{\text{st}})$ is the Drinfeld double Poisson Lie group of (G, π_{st}) (see §6.1), and $\Pi_{(G \times G)/(B \times B_-)}$ is the projection of Π_{st} to $(G \times G)/(B \times B_-)$. For $n \geq 1$, both F_n and \tilde{F}_n are quotient manifolds of G^n , and the Poisson structures π_n and $\tilde{\pi}_n$ are projections of the n -fold product Poisson structure π_{st}^n on G^n . Similarly, \mathbb{F}_n and $\tilde{\mathbb{F}}_n$ are quotient manifolds of $(G \times G)^n$, with Π_n and $\tilde{\Pi}_n$ projections of the product Poisson structure Π_{st}^n on $(G \times G)^n$. Precise definitions of the Poisson manifolds in (1.1) are given in §1.2.

This paper, a sequel to [38], is the second of a series of papers devoted to a detailed study of the four series of Poisson manifolds in (1.1). In [38], we have identified the Poisson structures in (1.1) as *mixed product Poisson structures defined by quasitriangular r -matrices*. In the present paper, we develop a general theory on \mathbb{T} -leaves and \mathbb{T} -leaf stabilizers for a class of Poisson structures defined by quasitriangular r -matrices and apply the theory to the Poisson manifolds in (1.1). The general theory also provides a unified approach to many other Poisson structures such as those mentioned earlier from [15, 16, 20, 35, 36, 39] (see §6.4).

Our descriptions of T -leaves for the Poisson manifolds in (1.1) naturally lead to what we call *extended Bruhat cells*, *extended Richardson varieties*, and *extended double Bruhat cells associated to conjugacy classes* (see §1.3 and §1.4). In [13], the third in the series of papers on the Poisson manifolds in (1.1), we express explicitly in the so-called Bott-Samelson coordinates the Poisson structures π_n on extended Bruhat cells in terms of the root strings and the structure constants of the Lie algebra \mathfrak{g} of G . In particular, we show in [13] that each extended Bruhat cell of dimension m gives rise to a polynomial

Poisson algebra $\mathbb{C}[z_1, \dots, z_m]$ which is a *symmetric nilpotent semi-quadratic Poisson-Ore extension* of \mathbb{C} in the sense of [21, Definition 4]. Moreover, when the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on \mathfrak{g} that comes into the definition of π_{st} is suitably chosen, the Poisson bracket $\{z_i, z_j\}$ between any two coordinate functions is in fact a polynomial with *integer* coefficients. In separate papers, we will further study extended Bruhat cells and extended double Bruhat cells in the context of symplectic groupoids.

Let \mathbb{T} be an algebraic torus. The \mathbb{T} -leaves in a \mathbb{T} -Poisson manifold (Y, π, λ) are the semi-classical analogs of \mathbb{T} -prime ideals of a quantum algebra A with rational \mathbb{T} -actions by automorphisms [6], and the \mathbb{T} -leaf decomposition of (Y, π, λ) is the semi-classical analog of the Goodearl-Letzler partition of the spectrum $\text{Spec}(A)$ of A into tori indexed by \mathbb{T} -invariant prime ideals [19]. In particular, if a \mathbb{T} -invariant prime ideal I in A corresponds to a \mathbb{T} -leaf L of (Y, π, λ) , the torus in the Goodearl-Letzler partition of $\text{Spec}(A)$ indexed by I should correspond to the quotient torus \mathbb{T}/\mathbb{T}_L , where \mathbb{T}_L is the sub-torus of \mathbb{T} preserving the symplectic leaves in L . In the case of the Bruhat cell $BuB/B \subset G/B$ with the Poisson structure $\pi_1 = \pi_{G/B}$, where $u \in W$, one has the quantum algebra \mathcal{U}_-^u constructed by De Concini, Kac, and Procesi [9] as a quantization of the algebra of regular functions on BuB/B (see [49]), and the explicit correspondence between the Goodearl-Letzler partition of $\text{Spec}(\mathcal{U}_-^u)$ and the T -leaves of $\pi_{G/B}$ in BuB/B , namely the open Richardson varieties $(BuB/B) \cap (B_-wB/B)$, $w \leq u$, have been studied in detail in [41, 50, 49, 51]. Similar studies for $(\tilde{F}_1, \tilde{\pi}_1) = (G, \pi_{\text{st}})$ can be found in [24, 26]. It would thus be very interesting to study the quantizations of the four series of Poisson manifolds in (1.1) (or of their Poisson submanifolds) and establish explicit correspondences between the Goodearl-Letzler partitions of the spectra of the quantizations and the T -leaf decompositions and the leaf stabilizers described in the current paper.

As the Poisson manifolds treated in this paper are special classes of Poisson homogeneous spaces, the general theory established in the paper can also be regarded a further development of Drinfeld's theory on Poisson homogeneous spaces [12, 39]. In particular, a generalization of Drinfeld's Lagrangian subalgebras associated to points in a Poisson homogeneous space [12] plays an important role in our general theory (see Lemma 3.16 and formulas (4.8) and (4.16) for detail).

We now give an outline and the main results of the paper.

1.2. Holomorphic Poisson structures related to flag varieties. If G is a group and $n \geq 1$ an integer, let the product group G^n act on itself from the right by

$$(1.2) \quad (g_1, g_2, \dots, g_n) \cdot (h_1, h_2, \dots, h_n) = (g_1 h_1, h_1^{-1} g_2 h_2, \dots, h_{n-1}^{-1} g_n h_n), \quad g_j, h_j \in G.$$

For subgroups Q_1, \dots, Q_n of G and subsets S_1, \dots, S_n of G such that S_1 is right Q_1 -invariant and S_j is left Q_{j-1} and right Q_j -invariant for $j = 2, \dots, n$, let $S_1 \times_{Q_1} \dots \times_{Q_{n-1}} S_n / Q_n$ denote the quotient of $S_1 \times \dots \times S_n$ by the action of $Q_1 \times \dots \times Q_n$ as a subgroup of G^n .

If (G, π_G) is a Poisson Lie group and if Q_1, \dots, Q_n are closed Poisson Lie subgroups of (G, π_G) , then the product Poisson structure π_G^n on G^n projects to a well-defined Poisson structure on the quotient space $G \times_{Q_1} \dots \times_{Q_{n-1}} G / Q_n$ (see [38, §7]). Throughout the paper, if $G \times_{Q_1} \dots \times_{Q_{n-1}} G / Q_n$ is denoted by Z_n , we will denote by π_{Z_n} the projection of π_G^n to Z_n and also refer to π_{Z_n} as a *quotient Poisson structure*. Denote the image $(g_1, \dots, g_n) \in G^n$ in Z_n by $[g_1, \dots, g_n]_{Z_n}$, and define

$$\mu_{Z_n} : Z_n \longrightarrow G/Q_n, \quad [g_1, g_2, \dots, g_n]_{Z_n} \longmapsto g_1 g_2 \dots g_n Q_n \in G/Q_n.$$

Then the map $\mu_{Z_n} : (Z_n, \pi_{Z_n}) \rightarrow (G/Q_n, \pi_{G/Q_n})$ is Poisson, and the action

$$(1.3) \quad G \times Z_n \longrightarrow Z_n, \quad (g, [g_1, g_2, \dots, g_n]_{Z_n}) \longmapsto [g g_1, g_2, \dots, g_n]_{Z_n}, \quad g, g_j \in G,$$

is a Poisson action of the Poisson Lie group (G, π_G) on the Poisson manifold (Z_n, π_{Z_n}) . This class of quotient Poisson structures was introduced in [38].

Let (G, π_{st}) be a standard complex semi-simple Poisson Lie group, determined by the choice of a pair (B, B_-) of opposite Borel subgroups of G and a symmetric non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on \mathfrak{g} . Let $(G \times G, \Pi_{\text{st}})$ be its Drinfeld double (see §6.1). Both B and B_- are Poisson Lie subgroups of (G, π_{st}) , while $B \times B_-$ is a Poisson Lie subgroup of $(G \times G, \Pi_{\text{st}})$. For an integer $n \geq 1$, let

$$(1.4) \quad F_n = G \times_B \cdots \times_B G/B, \quad \mathbb{F}_n = (G \times G) \times_{(B \times B_-)} \cdots \times_{(B \times B_-)} (G \times G)/(B \times B_-),$$

$$(1.5) \quad \tilde{F}_n = G \times_B \cdots \times_B G, \quad \tilde{\mathbb{F}}_n = (G \times G) \times_{(B \times B_-)} \cdots \times_{(B \times B_-)} (G \times G),$$

and let π_n and $\tilde{\pi}_n$ be the projections of π_{st}^n from G^n to F_n and \tilde{F}_n respectively, and let Π_n and $\tilde{\Pi}_n$ be the projections of Π_{st}^n from $(G \times G)^n$ to \mathbb{F}_n and $\tilde{\mathbb{F}}_n$ respectively. The maximal torus $T = B \cap B_-$ of G acts on (F_n, π_n) , (\mathbb{F}_n, Π_n) , $(\tilde{F}_n, \tilde{\pi}_n)$, and $(\tilde{\mathbb{F}}_n, \tilde{\Pi}_n)$ by Poisson diffeomorphisms via

$$(1.6) \quad t \cdot [g_1, g_2, \dots, g_n]_{F_n} = [tg_1, g_2, \dots, g_n]_{F_n},$$

$$(1.7) \quad t \cdot [g_1, k_1, g_2, k_2, \dots, g_n, k_n]_{\mathbb{F}_n} = [tg_1, tk_1, g_2, k_2, \dots, g_n, k_n]_{\mathbb{F}_n},$$

$$(1.8) \quad t \cdot [g_1, g_2, \dots, g_n]_{\tilde{F}_n} = [tg_1, g_2, \dots, g_n]_{\tilde{F}_n},$$

$$(1.9) \quad t \cdot [g_1, k_1, g_2, k_2, \dots, g_n, k_n]_{\tilde{\mathbb{F}}_n} = [tg_1, tk_1, g_2, k_2, \dots, g_n, k_n]_{\tilde{\mathbb{F}}_n},$$

where $t \in T$ and $g_j, k_j \in G$ for $1 \leq j \leq n$. Let \mathfrak{h} be the Lie algebra of T . For $Z_n \in \{F_n, \mathbb{F}_n, \tilde{F}_n, \tilde{\mathbb{F}}_n\}$, let $\lambda_{Z_n} : \mathfrak{h} \rightarrow \mathcal{V}^1(Z_n)$ be the Lie algebra action of \mathfrak{h} on Z_n generated by the action of T on Z_n , and for $z \in Z_n$, define the *leaf stabilizer* of λ_{Z_n} at z to be

$$\mathfrak{t}_z = \{x \in \mathfrak{h} : \lambda_{Z_n}(x)(z) \in T_z \Sigma_z\},$$

where Σ_z is the symplectic leaf of π_{Z_n} in Z_n through z .

Let $W = N_G(T)/T$ be the Weyl group of (G, T) , where $N_G(T)$ is the normalizer of T in G . Let \leq be the Bruhat order on W , and recall the monoidal product $*$ on W defined by

$$\overline{BuBvB} = \overline{B(u*v)B}, \quad u, v \in W,$$

where for a subset X of G , \overline{X} denotes the Zariski closure of X in G . For $\mathbf{u} = (u_1, \dots, u_n) \in W^n$, let $l(\mathbf{u}) = l(u_1) + \dots + l(u_n)$, where $l : W \rightarrow \mathbb{N}$ is the length function on W , and let

$$BuB = (Bu_1B) \times_B \dots \times_B (Bu_nB) \subset \tilde{F}_n.$$

For another sequence $\mathbf{v} = (v_1, \dots, v_n) \in W^n$, let

$$(B \times B_-)(\mathbf{u}, \mathbf{v})(B \times B_-) = (Bu_1B \times B_-v_1B_-) \times_{(B \times B_-)} \cdots \times_{(B \times B_-)} (Bu_nB \times B_-v_nB_-) \subset \tilde{\mathbb{F}}_n.$$

The images of $BuB \subset \tilde{F}_n$ and $(B \times B_-)(\mathbf{u}, \mathbf{v})(B \times B_-) \subset \tilde{\mathbb{F}}_n$ under the projections

$$\tilde{F}_n \longrightarrow F_n, \quad [g_1, g_2, \dots, g_n]_{\tilde{F}_n} \longmapsto [g_1, g_2, \dots, g_n]_{F_n}, \quad g_j \in G,$$

$$\tilde{\mathbb{F}}_n \longrightarrow \mathbb{F}_n, \quad [g_1, k_1, g_2, k_2, \dots, g_n, k_n]_{\tilde{\mathbb{F}}_n} \longmapsto [g_1, k_1, g_2, k_2, \dots, g_n, k_n]_{\mathbb{F}_n}, \quad g_j, k_j \in G,$$

will be respectively denoted by $BuB/B \subset F_n$ and $(B \times B_-)(\mathbf{u}, \mathbf{v})(B \times B_-)/(B \times B_-) \subset \mathbb{F}_n$.

We now state our results on the T -leaves and leaf stabilizers for each one of the four series in (1.1).

The following Theorem 1.1 on (F_n, π_n) , $n \geq 1$, will be proved in §6.2. For $n = 1$, Parts 1) and 2) of Theorem 1.1 have been proved in [20, Theorem 0.4] and [51, Theorem 3.1].

Theorem 1.1. *For $\mathbf{u} = (u_1, \dots, u_n) \in W^n$ and $w \in W$, let*

$$R_w^{\mathbf{u}} = (BuB/B) \cap \mu_{F_n}^{-1}(B_-wB/B) \subset F_n \quad \text{and} \quad \mathfrak{h}_w^{\mathbf{u}} = \{x + u_1 \cdots u_n w^{-1}(x) : x \in \mathfrak{h}\} \subset \mathfrak{h}.$$

1) $R_w^{\mathbf{u}} \neq \emptyset$ if and only if $w \leq u_1 * \dots * u_n$, and in this case, $R_w^{\mathbf{u}}$ is a connected smooth submanifold of F_n with $\dim R_w^{\mathbf{u}} = l(\mathbf{u}) - l(w)$;

2) The decomposition of F_n into T -leaves of the Poisson structure π_n is

$$F_n = \bigsqcup_{\mathbf{u} \in W^n, w \in W} R_w^{\mathbf{u}},$$

and all the symplectic leaves of π_n in $R_w^{\mathbf{u}}$ have dimensions equal to

$$l(\mathbf{u}) - l(w) - \dim \ker(1 + u_1 u_2 \dots u_n w^{-1});$$

3) The leaf stabilizer of λ_{F_n} at $z \in R_w^{\mathbf{u}}$ is $\mathfrak{t}_z = \mathfrak{h}_w^{\mathbf{u}}$.

The following Theorem 1.2 on (\mathbb{F}_n, Π_n) , $n \geq 1$, will be proved in §6.2. For $n = 1$, Theorem 1.2 has been proved in [15, §4].

Theorem 1.2. For $\mathbf{u} = (u_1, \dots, u_n), \mathbf{v} = (v_1, \dots, v_n) \in W^n$, and $w \in W$, let

$$\begin{aligned} G(w) &= G_{\text{diag}}(w, e)(B \times B_-)/(B \times B_-) \subset (G \times G)/(B \times B_-), \\ R_w^{\mathbf{u}, \mathbf{v}} &= ((B \times B_-)(\mathbf{u}, \mathbf{v})(B \times B_-)/(B \times B_-)) \cap \mu_{\mathbb{F}_n}^{-1}(G(w)) \subset \mathbb{F}_n, \\ \mathfrak{h}_w^{\mathbf{u}, \mathbf{v}} &= \{x + u_1 \dots u_n w^{-1}(v_1 \dots v_n)^{-1}(x) : x \in \mathfrak{h}\} \subset \mathfrak{h}, \end{aligned}$$

where $G_{\text{diag}} = \{(g, g) : g \in G\}$.

1) $R_w^{\mathbf{u}, \mathbf{v}} \neq \emptyset$ if and only if $w \leq (v_1 * \dots * v_n)^{-1} * u_1 * \dots * u_n$, and in this case, $R_w^{\mathbf{u}, \mathbf{v}}$ is a connected smooth submanifold of \mathbb{F}_n with $\dim R_w^{\mathbf{u}, \mathbf{v}} = l(\mathbf{u}) + l(\mathbf{v}) - l(w)$;

2) The decomposition of \mathbb{F}_n into T -leaves of the Poisson structure Π_n is

$$\mathbb{F}_n = \bigsqcup_{\mathbf{u}, \mathbf{v} \in W^n, w \in W} R_w^{\mathbf{u}, \mathbf{v}},$$

and all the symplectic leaves of Π_n in $R_w^{\mathbf{u}, \mathbf{v}}$ have dimensions equal to

$$l(\mathbf{u}) + l(\mathbf{v}) - l(w) - \dim \ker(1 + u_1 \dots u_n w^{-1}(v_1 \dots v_n)^{-1});$$

3) The leaf stabilizer of $\lambda_{\mathbb{F}_n}$ at $z \in R_w^{\mathbf{u}, \mathbf{v}}$ is $\mathfrak{t}_z = \mathfrak{h}_w^{\mathbf{u}, \mathbf{v}}$.

The following Theorem 1.3 on $(\tilde{F}_n, \tilde{\pi}_n)$, $n \geq 1$, will be proved in §6.3. For $n = 1$, Parts 1) and 2) of Theorem 1.3 have been proved in [30].

Theorem 1.3. 1) For any $\mathbf{u} = (u_1, \dots, u_n) \in W$ and $v \in W$, the intersection $(B\mathbf{u}B) \cap \mu_{\tilde{F}_n}^{-1}(B_- v B_-)$ in \tilde{F}_n is a non-empty smooth connected submanifold of \tilde{F}_n of dimension $l(\mathbf{u}) + l(v) + \dim T$;

2) The decomposition of \tilde{F}_n into T -leaves of the Poisson structure $\tilde{\pi}_n$ is

$$\tilde{F}_n = \bigsqcup_{\mathbf{u} \in W^n, v \in W} (B\mathbf{u}B) \cap \mu_{\tilde{F}_n}^{-1}(B_- v B_-),$$

and all the symplectic leaves of $\tilde{\pi}_n$ in $(B\mathbf{u}B) \cap \mu_{\tilde{F}_n}^{-1}(B_- v B_-)$ have dimensions equal to

$$l(\mathbf{u}) + l(v) + \dim \text{Im}(1 - u_1 \dots u_n v^{-1});$$

3) The leaf stabilizer of $\lambda_{\tilde{F}_n}$ at $z \in (B\mathbf{u}B) \cap \mu_{\tilde{F}_n}^{-1}(B_- v B_-)$ is $\mathfrak{t}_z = \{x - u_1 \dots u_n v^{-1}(x) : x \in \mathfrak{t}\}$.

Let \mathcal{C} be the set of all conjugacy classes in G . The following Theorem 1.4 on $(\tilde{\mathbb{F}}_n, \tilde{\Pi}_n)$, $n \geq 1$, will be proved in §6.3. For $n = 1$, Theorem 1.4 has been proved in [36, §7.3].

Theorem 1.4. For $C \in \mathcal{C}$, let $\Omega_C = G_{\text{diag}} \cdot (C \times \{e\}) \cdot G_{\text{diag}} = \{(g, h) \in G \times G : gh^{-1} \in C\}$, and for $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n) \in W^n$, let

$$R_C^{\mathbf{u}, \mathbf{v}} = ((B \times B_-)(\mathbf{u}, \mathbf{v})(B \times B_-)) \cap \mu_{\tilde{\mathbb{F}}_n}^{-1}(\Omega_C) \subset \tilde{\mathbb{F}}_n,$$

$$\mathfrak{h}^{\mathbf{u}, \mathbf{v}} = \{x - u_1 \cdots u_n (v_1 \cdots v_n)^{-1}(x) : x \in \mathfrak{h}\} \subset \mathfrak{h}.$$

1) For any $\mathbf{u}, \mathbf{v} \in W^n$ and $C \in \mathcal{C}$, $R_C^{\mathbf{u}, \mathbf{v}}$ is a connected smooth submanifold of $\tilde{\mathbb{F}}_n$ of dimension $l(\mathbf{u}) + l(\mathbf{v}) + \dim C + \dim T$;

2) The decomposition of $\tilde{\mathbb{F}}_n$ into T -leaves of the Poisson structure $\tilde{\Pi}_n$ is

$$(1.10) \quad \tilde{\mathbb{F}}_n = \bigsqcup_{\mathbf{u}, \mathbf{v} \in W^n, C \in \mathcal{C}} R_C^{\mathbf{u}, \mathbf{v}},$$

and all symplectic leaves of $\tilde{\Pi}_n$ in $R_C^{\mathbf{u}, \mathbf{v}}$ have dimensions equal to

$$l(\mathbf{u}) + l(\mathbf{v}) + \dim C + \dim \text{Im}(1 - u_1 \cdots u_n (v_1 \cdots v_n)^{-1});$$

3) The leaf stabilizer of $\lambda_{\tilde{\mathbb{F}}_n}$ at $z \in R_C^{\mathbf{u}, \mathbf{v}}$ is $\mathfrak{t}_z = \mathfrak{h}^{\mathbf{u}, \mathbf{v}}$.

1.3. Extended Bruhat cells, Bott-Samelson varieties, and extended Richardson varieties.

Let $n \geq 1$ and consider the disjoint union

$$F_n = \bigsqcup_{\mathbf{u} \in W^n} B\mathbf{u}B/B.$$

For $\mathbf{u} = (u_1, \dots, u_n) \in W^n$, we will call

$$B\mathbf{u}B/B = (Bu_1B \times_B \cdots \times_B Bu_nB)/B \subset F_n$$

an *extended Bruhat cell*. By Theorem 1.1, extended Bruhat cells in F_n are Poisson submanifolds with respect to the Poisson structure π_n . If $\mathbf{u} = (s_1, \dots, s_n)$, where each $s_j \in W$ is a simple reflection, we say that the extended Bruhat cell $B\mathbf{u}B/B$ in F_n is of *Bott-Samelson type*.

Every extended Bruhat cell $B\mathbf{u}B/B \subset F_n$ with the Poisson structure π_n is Poisson isomorphic to an extended Bruhat cell of Bott-Samelson type in $F_{l(\mathbf{u})}$ with the Poisson structure $\pi_{l(\mathbf{u})}$. Indeed, consider first the case of $n = 1$: if $u \in W$ and $u = s_1 \cdots s_{l(u)}$ is a reduced decomposition of u , the Poisson morphism $\mu_{F_{l(u)}} : (F_{l(u)}, \pi_{l(u)}) \rightarrow (G/B, \pi_1)$ then restricts to a Poisson isomorphism

$$(Bs(u)B/B, \pi_{l(u)}) \longrightarrow (BuB/B, \pi_1),$$

where $\mathbf{s}(u) = (s_1, \dots, s_{l(u)})$. For any arbitrary $\mathbf{u} = (u_1, \dots, u_n) \in W^n$, the choice of a reduced word $\mathbf{s}(u_j) \in W^{l(u_j)}$ for each u_j gives rise to a Poisson isomorphism from $(B\mathbf{u}B/B, \pi_n)$ to the Bott-Samelson type extended Bruhat cell $B(\mathbf{s}(u_1), \dots, \mathbf{s}(u_n))B/B$ in $F_{l(\mathbf{u})}$.

On the other hand, recall that for any sequence (s_1, \dots, s_n) of simple reflections in W^n , the Bott-Samelson variety $Z_{(s_1, \dots, s_n)}$ can be defined as the Zariski closure of $B(s_1, \dots, s_n)B/B$ in F_n and thus carries the Poisson structure π_n . Bott-Samelson varieties, with the Poisson structures thus defined, are therefore the building blocks for the Poisson manifolds (F_n, π_n) , $n \geq 1$, and are important even for the study of (F_1, π_1) . In [13], a sequel to the current paper, the Poisson structure π_n on any Bott-Samelson variety $Z_{(s_1, \dots, s_n)}$ is explicitly computed in each of the 2^n natural affine coordinate charts. In particular, it is shown in [13] that in each of these affine coordinate charts, π_n gives rise to a polynomial Poisson algebra $\mathbb{C}[z_1, \dots, z_n]$ which is a Poisson-Ore extension of \mathbb{C} compatible with the T -action. When the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is such that $\frac{1}{2}\langle \alpha, \alpha \rangle_{\mathfrak{g}} \in \mathbb{Z}$ for every root of \mathfrak{g} , it is shown in [13] that the Poisson structure on $\mathbb{C}[z_1, \dots, z_n]$ corresponding to each of the 2^n affine coordinate charts has the property that the Poisson brackets $\{z_i, z_k\}$ between the coordinate functions are polynomials with coefficients in \mathbb{Z} , thus giving rise to a Poisson-Ore extension of any field \mathbf{k} of arbitrary characteristic.

For $\mathbf{u} = (u_1, \dots, u_n) \in W^n$ and $w \in W$ such that $w \leq u_1 * \dots * u_n$, it is natural to refer to

$$R_w^{\mathbf{u}} = (B\mathbf{u}B/B) \cap \mu_{F_n}^{-1}(B_-wB/B) \subset F_n$$

as an (open) *extended Richardson variety* and its closure $\overline{R_w^{\mathbf{u}}}$ in F_n an *extended Richardson variety* in F_n . By Theorem 1.1, extended Richardson varieties in F_n are precisely closures of T -leaves of the Poisson structure π_n in F_n . It would be very interesting to extend the work of Lenagan and Yakimov in [33] on cluster structures on Richardson varieties in $F_1 = G/B$ to extended Richardson varieties.

Analogous to taking Zariski closures in F_n of extended Bruhat cells of Bott-Samelson type, one can consider the Zariski closures in \mathbb{F}_n of $(B \times B_-)(\mathbf{u}, \mathbf{v})(B \times B_-)/(B \times B_-)$, where $\mathbf{u}, \mathbf{v} \in W^n$ are sequences of simple reflections. Such closures are examples of *double Bott-Samelson varieties* introduced in [43], and carry the Poisson structure Π_n . Some combinatorial aspects of double Bott-Samelson varieties and calculations of the Poisson structure Π_n in special coordinate charts have been given in [42, 43].

1.4. Extended double Bruhat cells associated to conjugacy classes. Analogous to the Poisson manifold $(\tilde{F}_n = G \times_B \dots \times_B G, \tilde{\pi}_n)$, $n \geq 1$, one also has the quotient manifold

$$\tilde{F}_{-n} = G \times_{B_-} \dots \times_{B_-} G$$

of G^n with the well-defined Poisson structure $\tilde{\pi}_{-n}$, the projection of the Poisson structure π_{st}^n from G^n to \tilde{F}_{-n} . Consider, on the other hand, the diffeomorphism $S_{\tilde{\mathbb{F}}_n} : \tilde{\mathbb{F}}_n \rightarrow \tilde{F}_n \times \tilde{F}_{-n}$ given by

$$[g_1, k_1, \dots, g_n, k_n]_{\tilde{\mathbb{F}}_n} \mapsto ([g_1, \dots, g_n]_{\tilde{F}_n}, [k_1, \dots, k_n]_{\tilde{F}_{-n}}),$$

and set $\tilde{\pi}_{n,n} = S_{\tilde{\mathbb{F}}_n}^*(\tilde{\Pi}_n)$. By a result in [38, §8], $\tilde{\pi}_{n,n}$ is a *two-fold mixed product Poisson structure* on the product manifold $\tilde{F}_n \times \tilde{F}_{-n}$, i.e., it is the sum of the product Poisson structure $\tilde{\pi}_n \times \tilde{\pi}_{-n}$ and a certain mixed term defined by the action of B on \tilde{F}_n and B_- on \tilde{F}_{-n} by left translation. For $(\mathbf{u}, \mathbf{v}) = (u_1, \dots, u_n, v_1, \dots, v_n) \in W^{2n}$ and any conjugacy class C in G , define

$$\begin{aligned} G_C^{\mathbf{u}, \mathbf{v}} &= \{([g_1, \dots, g_n]_{\tilde{F}_n}, [k_1, \dots, k_n]_{\tilde{F}_{-n}}) \in (B\mathbf{u}B) \times (B_- \mathbf{v} B_-) : g_1 g_2 \dots g_n (k_1 k_2 \dots k_n)^{-1} \in C\} \\ &\subset \tilde{F}_n \times \tilde{F}_{-n}, \end{aligned}$$

where $(B_- \mathbf{v} B_-) = (B_- v_1 B_-) \times_{B_-} \dots \times_{B_-} (B_- v_n B_-) \subset \tilde{F}_{-n}$. By Theorem 1.4, each $G_C^{\mathbf{u}, \mathbf{v}}$ is a T -leaf of $\tilde{\pi}_{n,n}$ in $\tilde{F}_n \times \tilde{F}_{-n}$ for the diagonal action of T . We call $G_C^{\mathbf{u}, \mathbf{v}}$ an *extended double Bruhat cell associated to the conjugacy class C* . The case of $n = 1$ has been considered in [36, §7.3], where $G_C^{u,v} \subset G$, for $u, v \in W$ and a conjugacy class C in G , is called a *double Bruhat cell associated to the conjugacy class C* . Note that for $\mathbf{u}, \mathbf{v} \in W^n$ and $C = \{e\}$, we have

$$G_{\{e\}}^{\mathbf{u}, \mathbf{v}} := G_{\{e\}}^{\mathbf{u}, \mathbf{v}} = \{([g_1, \dots, g_n]_{\tilde{F}_n}, [k_1, \dots, k_n]_{\tilde{F}_{-n}}) \in (B\mathbf{u}B) \times (B_- \mathbf{v} B_-) : g_1 g_2 \dots g_n = k_1 k_2 \dots k_n\},$$

a direct generalization of double Bruhat cells in G . In a separate paper, we will study extended Bruhat cells and extended double Bruhat cells via Poisson groupoids and symplectic groupoids.

1.5. General theory. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a quasitriangular r -matrix on a Lie algebra \mathfrak{g} (definition recalled in §3.1), and let $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ be a Lie algebra action of \mathfrak{g} on a manifold Y . A simple observation made in [38] (see also [34, §2.1]) is that if the 2-tensor field $\lambda(r)$ on Y is skew-symmetric, then it is Poisson, and in this case we say that the Poisson structure $-\lambda(r)$ (or $\lambda(r)$) on Y is *defined* by the Lie algebra action λ and the quasitriangular r -matrix r . We also refer to Poisson structures obtained this way simply as *Poisson structures defined by quasitriangular r -matrices*.

Consider again the quotient manifold $Z_n = G \times_{Q_1} \dots \times_{Q_{n-1}} G / Q_n$ with the quotient Poisson structure π_{Z_n} for an arbitrary Poisson Lie group (G, π_G) and closed Poisson Lie subgroups Q_1, \dots, Q_n described

in §1.2. Note that the manifold Z_n is diffeomorphic to the product manifold $G/Q_1 \times \cdots \times G/Q_n$ via the diffeomorphism $J_{Z_n} : Z_n \rightarrow G/Q_1 \times \cdots \times G/Q_n$ given by

$$(1.11) \quad J_{Z_n}([g_1, g_2, \dots, g_n]_{Z_n}) = (g_1 Q_1, g_1 g_2 Q_2, \dots, g_1 g_2 \cdots g_n Q_n), \quad g_1, \dots, g_n \in G.$$

A key fact established in [38, §7] is that the Poisson structure $J_{Z_n}(\pi_{Z_n})$ on $G/Q_1 \times \cdots \times G/Q_n$ is defined by a quasitriangular r -matrix (see §5.3).

After a review on the notion of \mathbb{T} -leaves in §2 and on Poisson structures defined by quasitriangular r -matrices in §3, we devote §4 - §5 of the paper to a general theory on \mathbb{T} -leaves and \mathbb{T} -leaf stabilizers for a class of \mathbb{T} -invariant Poisson structures defined by quasitriangular r -matrices. Theorem 1.1 - Theorem 1.4 are proved in §6 as immediate examples of the general theory.

To give an outline of the general theory, let G be a connected Lie group with Lie algebra \mathfrak{g} and $r \in \mathfrak{g} \otimes \mathfrak{g}$ a factorizable quasitriangular r -matrix on \mathfrak{g} . The symmetric part of r determines a symmetric non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on \mathfrak{g} (see §3.2). If Y is a manifold with a Lie group action λ by G , we say that the quadruple (G, r, Y, λ) is *strongly admissible* if the stabilizer subgroup Q_y of G at every $y \in Y$ is connected and its Lie algebra \mathfrak{q}_y satisfies

$$[\mathfrak{q}_y, \mathfrak{q}_y] \subset \mathfrak{q}_y^{\perp} \subset \mathfrak{q}_y,$$

where $\mathfrak{v}^{\perp} = \{x \in \mathfrak{g} : \langle x, \mathfrak{v} \rangle_{\mathfrak{g}} = 0\}$ for a subspace \mathfrak{v} of \mathfrak{g} . If (G, r, Y, λ) is strongly admissible, the 2-tensor field $\lambda(r)$ on Y is necessarily skew-symmetric (see §3.3), so it is a Poisson structure on Y . Associated to the factorizable quasitriangular r -matrix r one also has two distinguished Lie subalgebras $\mathfrak{f}_+ = \text{Im}(r_+^b)$ and $\mathfrak{f}_- = \text{Im}(r_-^b)$ of \mathfrak{g} , where if $r = \sum_i x_i \otimes x'_i \in \mathfrak{g} \otimes \mathfrak{g}$, then $r_+^b, r_-^b : \mathfrak{g} \rightarrow \mathfrak{g}$ are respectively given by

$$r_+^b(x) = \sum_i \langle x, x_i \rangle_{\mathfrak{g}} x'_i \quad \text{and} \quad r_-^b(x) = - \sum_i \langle x, x'_i \rangle_{\mathfrak{g}} x_i, \quad x \in \mathfrak{g}.$$

A pair (M_+, M_-) of connected Lie subgroups of G is said to be *r -admissible* if their respective Lie algebras \mathfrak{m}_+ and \mathfrak{m}_- satisfy

$$\mathfrak{f}_+ \subset \mathfrak{m}_+, \quad \mathfrak{f}_- \subset \mathfrak{m}_-, \quad [\mathfrak{m}_+, \mathfrak{m}_+] \subset \mathfrak{m}_+^{\perp}, \quad [\mathfrak{m}_-, \mathfrak{m}_-] \subset \mathfrak{m}_-^{\perp}.$$

In §4 we consider a six-tuple $(G, r, Y, \lambda, M_+, M_-)$, where (G, r, Y, λ) is a strongly admissible quadruple, and (M_+, M_-) is a pair of r -admissible Lie subgroups of G . Let \mathbb{T} be the connected component of $M_+ \cap M_-$ containing the identity element. Then \mathbb{T} is necessarily abelian and acts on $(Y, \lambda(r))$ through λ by Poisson isomorphisms. On the other hand, one has the disjoint union

$$(1.12) \quad Y = \bigsqcup_{\mathcal{O}_+, \mathcal{O}_-} \mathcal{O}_+ \cap \mathcal{O}_-,$$

where \mathcal{O}_+ and \mathcal{O}_- are respectively M_+ -orbits and M_- -orbits in Y . The conditions $\mathfrak{f}_+ \subset \mathfrak{m}_+$ and $\mathfrak{f}_- \subset \mathfrak{m}_-$ imply that each non-empty intersection $\mathcal{O}_+ \cap \mathcal{O}_-$ in (1.12) is a \mathbb{T} -invariant Poisson submanifold of the \mathbb{T} -Poisson manifold $(Y, \lambda(r), \lambda)$. We say that the six-tuple $(G, r, Y, \lambda, M_+, M_-)$ is *admissible* if the \mathbb{T} -leaves of the Poisson structure $\lambda(r)$ in Y are precisely the connected components of all the non-empty intersections $\mathcal{O}_+ \cap \mathcal{O}_-$ in (1.12).

For each pair $(\mathcal{O}_+, \mathcal{O}_-)$ of M_+ - and M_- -orbits contained in the same G -orbit in Y , we introduce an integer $\delta_{\mathcal{O}_+, \mathcal{O}_-}$, given in (4.8), and a subspace $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$ of the Lie algebra $\mathfrak{t} = \mathfrak{m}_+ \cap \mathfrak{m}_-$ of \mathbb{T} , given in (4.16). Both $\delta_{\mathcal{O}_+, \mathcal{O}_-}$ and $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$ are defined using arbitrary points $y_+ \in \mathcal{O}_+$ and $y_- \in \mathcal{O}_-$ but are independent of the choices. The main results of §4 are summarized in the following Theorem 1.5.

Theorem 1.5. *Suppose that $\delta_{\mathcal{O}_+, \mathcal{O}_-} = 0$ for every pair $(\mathcal{O}_+, \mathcal{O}_-)$ of M_+ - and M_- -orbits. Then the six-tuple $(G, r, Y, \lambda, M_+, M_-)$ is admissible; the leaf stabilizer of each \mathbb{T} -leaf in $\mathcal{O}_+ \cap \mathcal{O}_-$ is $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$, and the co-rank of the Poisson structure $\lambda(r)$ in $\mathcal{O}_+ \cap \mathcal{O}_-$ is equal to the co-dimension of $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$ in \mathfrak{t} .*

The second part of the general theory, presented in §5, is an application of the “test” described in Theorem 1.5: assume that Q is a closed and connected Lie subgroup of G whose Lie algebra satisfies $\mathfrak{f}_+ \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}^\perp$. Equip G with the Poisson structure $\pi_G = r^L - r^R$, where r^L (resp. r^R) is the left (resp. right) invariant tensor field on G with value r at the identity element of G . Then [38, §7] Q is a Poisson Lie subgroup of the Poisson Lie group (G, π_G) , and one thus has the quotient Poisson structure π_{Z_n} on the quotient manifold $Z_n = G \times_Q \cdots \times_Q G/Q$, as explained in §1.2. Let (M_+, M_-) be again an r -admissible pair of Lie subgroups of G with respective Lie algebras \mathfrak{m}_+ and \mathfrak{m}_- . The connected component \mathbb{T} of $M_+ \cap M_-$ containing the identity element then acts on (Z_n, π_{Z_n}) by Poisson isomorphisms via (1.3). We study the \mathbb{T} -leaves of π_{Z_n} in Z_n via the \mathbb{T} -leaves of the Poisson structure $J_{Z_n}(\pi_{Z_n})$ on the n -fold product manifold $(G/Q)^n$, where $J_{Z_n} : Z_n \rightarrow (G/Q)^n$ is the diffeomorphism given in (1.11), and \mathbb{T} acts on $(G/Q)^n$ diagonally.

More precisely, let λ be the Lie algebra action of the direct product Lie algebra \mathfrak{g}^n on $(G/Q)^n$ induced from the left action of G^n on $(G/Q)^n$ by left translation. By a result in [38, §8] (see §5.3 for more detail), the Poisson structure $J_{Z_n}(\pi_{Z_n})$ on $(G/Q)^n$ is defined by the Lie algebra action λ of \mathfrak{g}^n and a certain quasitriangular r -matrix $r^{(n)}$ on \mathfrak{g}^n . One can thus study the \mathbb{T} -leaves of $J_{Z_n}(\pi_{Z_n})$ in $(G/Q)^n$ via the six-tuple

$$(1.13) \quad \left(G^n, r^{(n)}, (G/Q)^n, \lambda, M_+^{(n)}, M_-^{(n)} \right),$$

where $(M_+^{(n)}, M_-^{(n)})$ is an $r^{(n)}$ -admissible pair of Lie subgroups of G^n determined by (M_+, M_-) . Applying Theorem 1.5, we obtain in Proposition 5.5 sufficient conditions for the six-tuple in (1.13) to be admissible. Translating to (Z_n, π_{Z_n}) using the Poisson diffeomorphism

$$J_{Z_n} : (Z_n, \pi_{Z_n}) \longrightarrow ((G/Q)^n, J_{Z_n}(\pi_{Z_n})),$$

we obtain in Theorem 5.8 a description of \mathbb{T} -leaves and leaf stabilizers for (Z_n, π_{Z_n}) under some sufficient conditions on the triple (M_+, M_-, Q) . Theorem 1.1 and Theorem 1.2 are then proved in §6.2 as special cases of Theorem 5.8. Theorem 1.3 and Theorem 1.4 are similarly proved in §6.3 as special cases of Theorem 5.10, an analog of Theorem 5.8.

1.6. Notation. Throughout the paper, the pairing between a finite dimensional vector space V (over \mathbb{R} or \mathbb{C}) and its dual space V^* will always be denoted by $\langle \cdot, \cdot \rangle$. The annihilator of a vector subspace U of V is, by definition, the subspace of V^* given by $U^0 = \{\xi \in V^* : \langle \xi, U \rangle = 0\}$. For each integer $k \geq 1$, $\wedge^k V$ is identified with the subspace of skew-symmetric elements in $V^{\otimes k}$, and for $v_1, \dots, v_k \in V$,

$$(1.14) \quad v_1 \wedge v_2 \wedge \cdots \wedge v_k = \sum_{\lambda \in S_k} \text{sign}(\lambda) v_{\lambda(1)} \otimes v_{\lambda(2)} \otimes \cdots \otimes v_{\lambda(k)} \in \wedge^k V \subset V^{\otimes k}.$$

For an element $r = \sum_i u_i \otimes v_i \in V \otimes V$, define $r^{21} = \sum_i v_i \otimes u_i \in V \otimes V$ and

$$(1.15) \quad r^\# : V^* \longrightarrow V, \quad r^\#(\xi) = \sum_i \langle \xi, u_i \rangle v_i, \quad \xi \in V^*.$$

Then $(r^\#)^* = (r^{21})^\# : V^* \rightarrow V$.

If \mathfrak{g} is a Lie algebra over \mathbb{R} (resp. \mathbb{C}), by a *left* Lie algebra action of \mathfrak{g} on a manifold (resp. complex manifold) Y we mean a Lie algebra anti-homomorphism $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$, where $\mathcal{V}^1(Y)$ is the space of smooth (resp. holomorphic) vector fields on Y . For $k \geq 1$ and $X = \sum x_{i_1} \otimes \cdots \otimes x_{i_k} \in \mathfrak{g}^{\otimes k}$, let $\lambda(X)$ be the k -tensor field on Y given by

$$\lambda(X) = \sum \lambda(x_{i_1}) \otimes \cdots \otimes \lambda(x_{i_k}).$$

When G is a connected Lie group with Lie algebra \mathfrak{g} and $\lambda : G \times Y \rightarrow Y, (g, y) \mapsto gy$, is a left Lie group action, we use λ to also denote the induced left action of \mathfrak{g} on Y , i.e.,

$$(1.16) \quad \lambda : \mathfrak{g} \longrightarrow \mathcal{V}^1(Y), \quad \lambda(x)(y) = \left. \frac{d}{dt} \right|_{t=0} \exp(tx)y, \quad x \in \mathfrak{g}, y \in Y.$$

1.7. Acknowledgments. We would like to thank Allen Knutson to whom goes the credit of Lemma 2.5. Work in this paper has been partially supported by the Research Grants Council of the Hong Kong SAR, China (GRF HKU 704310 and 703712).

2. POISSON ACTIONS AND \mathbb{T} -LEAVES

2.1. Regular and full Poisson actions by Lie bialgebras. Recall that a *Lie bialgebra* over the field $\mathbf{k} = \mathbb{C}$ or \mathbb{R} is a pair (\mathfrak{g}, δ) , where \mathfrak{g} is a Lie algebra over \mathbf{k} and $\delta : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$, called the *co-bracket*, is a linear map satisfying the co-Jacobi identity and the cocycle condition

$$(2.1) \quad \delta[x, y] = \text{ad}_x(\delta(y)) - \text{ad}_y(\delta(x)), \quad x, y \in \mathfrak{g}.$$

Given a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$, the dual map of δ is then a Lie bracket on the dual space \mathfrak{g}^* of \mathfrak{g} .

A *left Poisson action* of the Lie bialgebra (\mathfrak{g}, δ) on a Poisson manifold (Y, π) is a left Lie algebra action $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ such that

$$L_{\lambda(x)}\pi = \lambda(\delta(x)), \quad x \in \mathfrak{g},$$

where for $x \in \mathfrak{g}$, $L_{\lambda(x)}\pi$ is the Lie derivative of π in the direction of the vector field $\lambda(x)$. Here when \mathfrak{g} is a Lie bialgebra over \mathbb{C} , we understand (Y, π) to be a complex Poisson manifold, i.e., Y is a complex manifold and π a holomorphic Poisson structure on Y .

Lie bialgebras and Poisson actions of Lie bialgebras are the infinitesimal counterparts of Poisson Lie groups and Poisson actions of Poisson Lie groups, and we refer to [8, 10, 11, 14, 46, 47] for the basics of the theory. In this paper we follow the notation and sign conventions in the review section §2 of [38].

Let $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ be a left Poisson action of a Lie bialgebra (\mathfrak{g}, δ) on a Poisson manifold (Y, π) . For $y \in Y$, set $\pi_y = \pi(y) \in \wedge^2 T_y Y$, and

$$\lambda_y : \mathfrak{g} \longrightarrow T_y Y : \lambda_y(x) = \lambda(x)(y), \quad y \in Y, x \in \mathfrak{g}.$$

Recall that $\text{Im}(\pi_y^\#)$, where $\pi_y^\# : T_y^* Y \rightarrow T_y Y$ is defined using (1.15), is the tangent space to the symplectic leaf of π through y , and the rank of π at y is, by definition, $\dim(\text{Im}(\pi_y^\#))$. Set

$$(2.2) \quad \Phi_y = \lambda_y(\mathfrak{g}) + \text{Im}(\pi_y^\#) \subset T_y Y.$$

Definition 2.1. The Poisson action λ of (\mathfrak{g}, δ) on (Y, π) is said to be *regular* (resp. *full*) if $\dim \Phi_y$ is independent of $y \in Y$ (resp. $\Phi_y = T_y Y$ for all $y \in Y$).

Given a left Poisson action λ of a Lie algebra (\mathfrak{g}, δ) on a Poisson manifold (Y, π) , it is shown in [37] that the vector bundle $(Y \times \mathfrak{g}) \oplus T^* Y$, the direct product of the trivial vector bundle over Y with fiber \mathfrak{g} and the cotangent bundle $T^* Y$ of Y , has the structure of a Lie algebroid over Y with $-\lambda + \pi^\#$ as the anchor map. We denote this Lie algebroid by

$$(2.3) \quad A_\lambda = (Y \times \mathfrak{g}) \bowtie T^* Y$$

and refer to [37] for details. Thus the Poisson action λ is regular (resp. full) if and only if the Lie algebroid A_λ is regular (resp. transitive), i.e., the anchor map $-\lambda + \pi^\#$ of A_λ has constant rank (resp. everywhere surjective).

For a Poisson action λ of a Lie bialgebra (\mathfrak{g}, δ) on (Y, π) , and for $y \in Y$, we also define

$$(2.4) \quad \mathfrak{g}_y = \{x \in \mathfrak{g} : \lambda_y(x) \in \text{Im}(\pi_y^\#)\} \subset \mathfrak{g}.$$

The linear map $\lambda_y : \mathfrak{g} \rightarrow T_y Y$ then induces a well-defined vector space isomorphism

$$(2.5) \quad \mathfrak{g}/\mathfrak{g}_y \longrightarrow \Phi_y/\text{Im}(\pi_y^\#) : \quad x + \mathfrak{g}_y \longmapsto \lambda_y(x) + \text{Im}(\pi_y^\#), \quad x \in \mathfrak{g}.$$

2.2. \mathbb{T} -leaves and leaf stabilizers. In §2.2, let (Y, π) be a real (resp. complex) Poisson manifold and \mathbb{T} a connected real (resp. complex) abelian Lie group acting on Y by Poisson isomorphisms. Denote the action by $\lambda : \mathbb{T} \times Y \rightarrow Y$ and the Lie algebra of \mathbb{T} by \mathfrak{t} . Then λ is a Poisson action of the Lie bialgebra $(\mathfrak{t}, 0)$ on (Y, π) , where 0 denotes the zero map $\mathfrak{t} \rightarrow \wedge^2 \mathfrak{t}$. We will refer to the triple (Y, π, λ) as a \mathbb{T} -Poisson manifold. Let $A_\lambda = (Y \times \mathfrak{t}) \bowtie T^*Y$ be the Lie algebroid of over Y given in (2.3).

Recall (see, for example, [17, 40]) that the *orbits* of the Lie algebroid A_λ are the integral submanifolds of the distribution on Y (not necessarily of constant rank) defined as the image of the anchor map of A_λ , i.e., of the distribution $\bigsqcup_{y \in Y} (\lambda_y(\mathfrak{t}) + \text{Im}(\pi_y^\#))$ on Y .

Definition 2.2. Orbits of the Lie algebroid A_λ are called the \mathbb{T} -orbits of symplectic leaves, or simply the \mathbb{T} -leaves, of π in Y .

The following Lemma 2.3 justifies the terminology in Definition 2.2.

Lemma 2.3. *Let L be any \mathbb{T} -leaf of π in Y and let Σ be any symplectic leaf of π such that $L \cap \Sigma \neq \emptyset$. Then $\Sigma \subset L$ and the action map $\lambda : \mathbb{T} \times \Sigma \rightarrow L$ is a surjective submersion.*

Proof. As $\bigsqcup_{y \in Y} \text{Im}(\pi_y^\#)$ is a sub-distribution of $\bigsqcup_{y \in Y} (\lambda_y(\mathfrak{t}) + \text{Im}(\pi_y^\#))$, one has $\Sigma \subset L$. Let $t \in \mathbb{T}$ and $y \in \Sigma$. The differential of λ at (t, y) is the linear map

$$\lambda_* : T_{(t,y)}(\mathbb{T} \times \Sigma) \longrightarrow T_{ty} L, \quad (r_t(x), v_y) \longmapsto \lambda_{ty}(x) + (\lambda_t)_* v_y, \quad x \in \mathfrak{t}, v_y \in T_y \Sigma,$$

where for $x \in \mathfrak{t}$, $r_t(x) \in T_t \mathbb{T}$ is the right translate of x by t , and $(\lambda_t)_* : T_y Y \rightarrow T_{ty} Y$ is the differential at y of the map $\lambda_t : Y \rightarrow Y, y_1 \rightarrow ty_1, y_1 \in Y$. Since λ_t preserves π , $(\lambda_t)_*(T_y \Sigma) = \text{Im}(\pi_{ty}^\#)$. Thus the action map λ is a submersion, and $\mathbb{T}\Sigma := \bigsqcup_{t \in \mathbb{T}} t\Sigma$ is open in L . If Σ' is an other symplectic leaf of π contained in L , then either $\mathbb{T}\Sigma = \mathbb{T}\Sigma'$ or $\mathbb{T}\Sigma \cap \mathbb{T}\Sigma' = \emptyset$, and since L is connected, one must have $\mathbb{T}\Sigma = \mathbb{T}\Sigma'$. Thus λ is surjective.

Q.E.D.

Definition 2.4. For $y \in Y$, the subspace

$$\mathfrak{t}_y = \{x \in \mathfrak{t} : \lambda_y(x) \in \text{Im}(\pi_y^\#)\}$$

of \mathfrak{t} is called the \mathbb{T} -leaf stabilizer (or simply the leaf stabilizer) of λ at y .

The following Lemma 2.5 is due to A. Knutson through private communication.

Lemma 2.5. *If y and y' are in the same \mathbb{T} -leaf of (Y, π) , then $\mathfrak{t}_y = \mathfrak{t}_{y'}$.*

Proof. Let L be any \mathbb{T} -leaf of π in Y . Let $2r$ be the rank of π in L , and let π^r be the r 'th exterior product of π with itself. For $x \in \mathfrak{t}$, let $V_x = \lambda(x) \wedge \pi^r$, a $(2r+1)$ -vector field on L . As \mathbb{T} is abelian and acts on L by Poisson isomorphisms, V_x is \mathbb{T} -invariant. For any local (smooth or holomorphic) function f on L , if $X_f = \pi^\#(df)$ is the Hamiltonian vector field of f , then

$$L_{X_f} V_x = (L_{X_f} \lambda(x)) \wedge \pi^r = X_{\lambda(x)(f)} \wedge \pi^r = 0.$$

Thus V_x is also invariant under local Hamiltonian flows. It follows that the vanishing locus $Z(V_x)$ of V_x is open in L . As $Z(V_x)$ is also closed and as L is connected, V_x vanishes everywhere on L if it vanishes

at one point. But for any $y \in L$, $V_x(y) = 0$ if and only if $x \in \mathfrak{t}_y$. It follows that the subspace \mathfrak{t}_y of \mathfrak{t} is independent of $y \in L$.

Q.E.D.

Definition 2.6. For a \mathbb{T} -leaf L in Y , the subspace $\mathfrak{t}_L := \mathfrak{t}_y$ of \mathfrak{t} , where $y \in L$ is arbitrary, will be called the *leaf stabilizer of λ in L* .

Remark 2.7. (1) Note that by the vector space isomorphism in (2.5), the co-rank of π in L is the same as the co-dimension of \mathfrak{t}_L in \mathfrak{t} ;

(2) Let $L \subset Y$ be a \mathbb{T} -leaf in Y with $\dim L = l$, and let $2r$ be the rank of π in L . Let $\mathfrak{t}'_L \subset \mathfrak{t}$ be any vector space complement of \mathfrak{t}_L in \mathfrak{t} . Then $\dim \mathfrak{t}'_L = l - 2r$. For any $\xi \in \wedge^{l-2r} \mathfrak{t}'_L$, $\xi \neq 0$, the l -vector field

$$(2.6) \quad \eta_L = \lambda(\xi) \wedge \underbrace{\pi \wedge \cdots \wedge \pi}_r$$

on Y then restricts to a nowhere vanishing anti-canonical section of L . It is also clear that up to non-zero scalar multiples, the restriction of η_L to L is independent of the choices of the complement \mathfrak{t}'_L of \mathfrak{t}_L in \mathfrak{t} and of the non-zero element $\xi \in \wedge^{l-2r} \mathfrak{t}'_L$. The restriction to L of η_L in (2.6) is called a Poisson \mathbb{T} -Pfaffian of the Poisson structure π on L , a term suggested by M. Yakimov and A. Knutson. \diamond

3. POISSON STRUCTURES DEFINED BY QUASITRIANGULAR r -MATRICES

3.1. Quasitriangular r -matrices. We recall some basic facts on quasitriangular r -matrices and Lie bialgebras. Our references are [8, 10, 11, 14, 38].

Let \mathfrak{g} be a Lie algebra. Recall that the Classical Yang-Baxter operator for \mathfrak{g} is the map $\text{CYB} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ given by

$$\text{CYB}(r) = \sum_{i,j} ([x_i, x_j] \otimes y_i \otimes y_j + x_i \otimes [y_i, x_j] \otimes y_j + x_i \otimes x_j \otimes [y_i, y_j]), \quad \text{if } r = \sum_i x_i \otimes y_i.$$

A *quasitriangular r -matrix* on \mathfrak{g} is an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $r + r^{21} \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ and that $\text{CYB}(r) = 0$, where $(S^2 \mathfrak{g})^{\mathfrak{g}}$ is the space of \mathfrak{g} -invariant elements in $S^2 \mathfrak{g}$ with respect to the adjoint action. A quasitriangular r -matrix r on \mathfrak{g} is said to be *factorizable* if $r + r^{21} \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ is non-degenerate.

Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a quasitriangular r -matrix on \mathfrak{g} . Define

$$(3.1) \quad \delta_r : \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}, \quad \delta_r(x) = \text{ad}_x(r), \quad x \in \mathfrak{g}.$$

As $r + r^{21} \in (S^2 \mathfrak{g})^{\mathfrak{g}}$, δ_r takes values in $\wedge^2 \mathfrak{g}$, and (\mathfrak{g}, δ_r) is a Lie bialgebra. Define $r_{\pm} : \mathfrak{g}^* \rightarrow \mathfrak{g}$ by

$$(3.2) \quad r_+ = r^{\#}, \quad r_- = -(r^{21})^{\#} = -r_+^*$$

(see (1.15)). The Lie bracket on \mathfrak{g}^* , defined as the dual map of δ_r , is then given by

$$(3.3) \quad [\xi, \eta] = \text{ad}_{r_+(\xi)}^* \eta - \text{ad}_{r_-(\eta)}^* \xi = \text{ad}_{r_-(\xi)}^* \eta - \text{ad}_{r_+(\eta)}^* \xi, \quad \xi, \eta \in \mathfrak{g}^*,$$

where for $x \in \mathfrak{g}$ and $\zeta \in \mathfrak{g}^*$, the element $\text{ad}_x^* \zeta \in \mathfrak{g}^*$ is given by $\langle \text{ad}_x^* \zeta, y \rangle = \langle \zeta, [y, x] \rangle$ for $y \in \mathfrak{g}$. It is well-known ([14, Lecture 4], [44]) that both r_+ and r_- are Lie algebra homomorphisms. Set

$$(3.4) \quad \mathfrak{f}_+ = \text{Im}(r_+), \quad \mathfrak{f}_- = \text{Im}(r_-).$$

Then both \mathfrak{f}_+ and \mathfrak{f}_- are Lie subalgebras of \mathfrak{g} , and $\delta_r(\mathfrak{f}_{\pm}) \subset \wedge^2 \mathfrak{f}_{\pm}$. In other words, both \mathfrak{f}_+ and \mathfrak{f}_- are sub-Lie bialgebras of the Lie bialgebra (\mathfrak{g}, δ_r) . A different proof of the following Lemma 3.1 can be found in [38, §7.1].

Lemma 3.1. *Any Lie subalgebra \mathfrak{m} of \mathfrak{g} containing \mathfrak{f}_+ or \mathfrak{f}_- is a sub-Lie bialgebra of (\mathfrak{g}, δ_r) .*

Proof. Assume that $\mathfrak{m} \supset \mathfrak{f}_+$. One needs to show that \mathfrak{m}^0 , the annihilator of \mathfrak{m} in \mathfrak{g}^* , is a Lie ideal with respect to the Lie bracket on \mathfrak{g}^* given in (3.3). Let $\xi \in \mathfrak{m}^0$ and $\eta \in \mathfrak{g}^*$. Since $\mathfrak{m} \supset \mathfrak{f}_+$, one has $\mathfrak{m}^0 \subset \mathfrak{f}_+^0 = \ker r_-$, and it follows from (3.3) that $[\xi, \eta] = -\text{ad}_{r_+(\eta)}^* \xi$. As $r_+(\eta) \in \mathfrak{f}_+ \subset \mathfrak{m}$, one has $\text{ad}_{r_+(\eta)}^* \xi \in \mathfrak{m}^0$. Thus \mathfrak{m}^0 is a Lie ideal of \mathfrak{g}^* . The case when $\mathfrak{m} \supset \mathfrak{f}_-$ is proved similarly.

Q.E.D.

3.2. Factorizable quasitriangular r -matrices. Recall that a quadratic Lie algebra is a pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$, where \mathfrak{g} is a Lie algebra and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is a symmetric non-degenerate invariant bilinear form on \mathfrak{g} . Given a quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$, for any vector subspace \mathfrak{v} of \mathfrak{g} , let

$$(3.5) \quad \mathfrak{v}^\perp = \{x \in \mathfrak{g} : \langle x, \mathfrak{v} \rangle_{\mathfrak{g}} = 0\}.$$

By a *Lagrangian subalgebra* of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ we mean a Lie subalgebra \mathfrak{l} of \mathfrak{g} that is also Lagrangian with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, i.e., $\mathfrak{l}^\perp = \mathfrak{l}$. By a *Lagrangian splitting* of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ we mean a decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{u}'$ where both \mathfrak{u} and \mathfrak{u}' are Lagrangian subalgebras of \mathfrak{g} . The notion of quadratic Lie algebras with Lagrangian splittings is then equivalent to that of Manin triples [14, Lecture 4].

Given a Lie bialgebra $(\mathfrak{u}, \delta_{\mathfrak{u}})$, recall that the *Drinfeld double Lie algebra* of $(\mathfrak{u}, \delta_{\mathfrak{u}})$ is the quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$, where $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{u}^*$ as a vector space, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is the symmetric bilinear form on \mathfrak{g} given by

$$\langle x + \xi, y + \eta \rangle_{\mathfrak{g}} = \langle x, \eta \rangle + \langle \xi, y \rangle, \quad x, y \in \mathfrak{u}, \xi, \eta \in \mathfrak{u}^*,$$

and the Lie bracket on \mathfrak{g} is the unique one with respect to which the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is invariant and both $\mathfrak{u} \cong \mathfrak{u} \oplus 0$ and $\mathfrak{u}^* \cong 0 \oplus \mathfrak{u}^*$ are Lie subalgebras. The decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{u}^*$ is thus a Lagrangian splitting of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$. One also refers to $((\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}}), \mathfrak{u}, \mathfrak{u}^*)$ as the *Manin triple of the Lie bialgebra* $(\mathfrak{u}, \delta_{\mathfrak{u}})$. Conversely, a Lagrangian splitting $\mathfrak{g} = \mathfrak{u} + \mathfrak{u}'$ of a quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ gives rise to a Lie bialgebra $(\mathfrak{u}, \delta_{\mathfrak{u}})$, where $\delta_{\mathfrak{u}} : \mathfrak{u} \rightarrow \wedge^2 \mathfrak{u}$ is the map dual to the Lie bracket on \mathfrak{u}' , the latter being identified with \mathfrak{u}^* via the pairing between \mathfrak{u} and \mathfrak{u}' defined by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$, and the Manin triple $((\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}}), \mathfrak{u}, \mathfrak{u}')$ is isomorphic to the Manin triple of $(\mathfrak{u}, \delta_{\mathfrak{u}})$.

Assume now that r is a factorizable quasitriangular r -matrix on a Lie algebra \mathfrak{g} , i.e., the linear map

$$r_+ - r_- = (r + r^{21})^\# : \mathfrak{g}^* \longrightarrow \mathfrak{g}$$

is invertible. The symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on \mathfrak{g} given by

$$(3.6) \quad \langle x_1, x_2 \rangle_{\mathfrak{g}} = \langle (r_+ - r_-)^{-1} x_1, x_2 \rangle = \langle x_1, (r_+ - r_-)^{-1} x_2 \rangle, \quad x_1, x_2 \in \mathfrak{g}.$$

is then non-degenerate and invariant, making $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ into a quadratic Lie algebra. We will refer to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ as the *symmetric bilinear form on \mathfrak{g} associated to r* . Set

$$(3.7) \quad r_\pm^b = r_\pm \circ (r_+ - r_-)^{-1} : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

One thus has $r_+^b - r_-^b = \text{Id}_{\mathfrak{g}}$, and

$$\langle r_+^b(x_1), x_2 \rangle_{\mathfrak{g}} + \langle x_1, r_-^b(x_2) \rangle_{\mathfrak{g}} = 0, \quad x_1, x_2 \in \mathfrak{g}.$$

It also follows from the definitions that $\mathfrak{f}_\pm = \text{Im}(r_\pm^b)$ and that

$$(3.8) \quad \ker(r_+) = \mathfrak{f}_-^0, \quad \ker(r_-) = \mathfrak{f}_+^0, \quad \ker(r_+^b) = \mathfrak{f}_-^\perp, \quad \ker(r_-^b) = \mathfrak{f}_+^\perp.$$

In particular, if $x \in \mathfrak{f}_+^\perp$, then $x = r_+^b(x) - r_-^b(x) = r_+^b(x) \in \mathfrak{f}_+$, so $\mathfrak{f}_+^\perp \subset \mathfrak{f}_+$. Similarly, $\mathfrak{f}_-^\perp \subset \mathfrak{f}_-$.

Still assuming that r is factorizable, consider now the direct product Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ and the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}}$ on $\mathfrak{g} \oplus \mathfrak{g}$ given by

$$(3.9) \quad \langle (x_1, x_2), (x'_1, x'_2) \rangle_{\mathfrak{g} \oplus \mathfrak{g}} = \langle x_1, x'_1 \rangle_{\mathfrak{g}} - \langle x_2, x'_2 \rangle_{\mathfrak{g}}, \quad x_1, x_2, x'_1, x'_2 \in \mathfrak{g}.$$

One then has the Lagrangian splitting

$$(3.10) \quad \mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + \mathfrak{l}_r,$$

of the quadratic Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$, where $\mathfrak{g}_{\text{diag}} = \{(x, x) : x \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}$, and

$$(3.11) \quad \mathfrak{l}_r = \{(r_+(\xi), r_-(\xi)) : \xi \in \mathfrak{g}^*\} = \{(r_+^b(x), r_-^b(x)) : x \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}.$$

One checks that $\delta_r : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ coincides with the co-bracket on $\mathfrak{g} \cong \mathfrak{g}_{\text{diag}}$ induced by the Lagrangian splitting in (3.10). The assignment $\mathfrak{g} \otimes \mathfrak{g} \ni r \mapsto \mathfrak{l}_r \subset \mathfrak{g} \oplus \mathfrak{g}$ gives a one to one correspondence between the set of all factorizable quasitriangular r -matrices on \mathfrak{g} that have $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ as the associated symmetric bilinear form and the set of Lagrangian subalgebras \mathfrak{l} of $(\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$ such that $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + \mathfrak{l}$.

Example 3.2. Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ be any quadratic Lie algebra and let $\mathfrak{g} = \mathfrak{u} + \mathfrak{u}'$ be a Lagrangian splitting of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$. The quadratic Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$ (see (3.9)) has the Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + \mathfrak{l}$, where $\mathfrak{l} = \{(\xi, x) : \xi \in \mathfrak{u}', x \in \mathfrak{u}\} \subset \mathfrak{g} \oplus \mathfrak{g}$. The factorizable quasitriangular r -matrix $r_{(\mathfrak{u}, \mathfrak{u}')}$ on \mathfrak{g} such that $\mathfrak{l}_{r_{(\mathfrak{u}, \mathfrak{u}')}} = \mathfrak{l}$ is given by

$$(3.12) \quad r_{(\mathfrak{u}, \mathfrak{u}')} = \sum_{i=1}^m x_i \otimes \xi_i \in \mathfrak{g} \otimes \mathfrak{g},$$

where $\{x_i\}_{i=1}^m$ is any basis of \mathfrak{u} and $\{\xi_i\}_{i=1}^m$ the basis of \mathfrak{u}' such that $\langle x_i, \xi_j \rangle_{\mathfrak{g}} = \delta_{ij}$ for $1 \leq i, j \leq m$. We will call $r_{(\mathfrak{u}, \mathfrak{u}')}$ the r -matrix on \mathfrak{g} defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{u} + \mathfrak{u}'$ of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$. It is easy to see that the Lie subalgebras \mathfrak{f}_- and \mathfrak{f}_+ of \mathfrak{g} associated to $r_{(\mathfrak{u}, \mathfrak{u}')}$ (see definitions in (3.4)) are respectively given by $\mathfrak{f}_- = \mathfrak{u}$ and $\mathfrak{f}_+ = \mathfrak{u}'$. In particular, $\mathfrak{f}_- \cap \mathfrak{f}_+ = 0$. Conversely, let r be a factorizable quasitriangular r -matrix on \mathfrak{g} with $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ as the associated symmetric form. If $\mathfrak{f}_- \cap \mathfrak{f}_+ = 0$, then $\mathfrak{g} = \mathfrak{f}_- + \mathfrak{f}_+$ is a Lagrangian splitting and $r = r_{(\mathfrak{f}_-, \mathfrak{f}_+)}$. \diamond

Example 3.3. Let \mathfrak{g} be a complex simple Lie algebra, let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be any non-zero scalar multiple of the Killing form of \mathfrak{g} , and let $\langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}}$ be the bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$ given in (3.9). In [2], Belavin and Drinfeld classified all Lagrangian splittings of $(\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$ of the form $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + \mathfrak{l}$ and wrote down explicitly the corresponding factorizable quasitriangular r -matrices on \mathfrak{g} . Details on the so-called *standard r -matrix* r_{st} on \mathfrak{g} will be recalled in §6.1. \diamond

3.3. Poisson structures defined by quasitriangular r -matrices. Let Y be a manifold with a left Lie algebra action $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$. For $r \in \mathfrak{g} \otimes \mathfrak{g}$, let $\lambda(r)$ be the tensor field on Y given by

$$(3.13) \quad \lambda(r) = \sum_i \lambda(x_i) \otimes \lambda(y_i), \quad \text{if } r = \sum_i x_i \otimes y_i.$$

The special case of the following Lemma 3.4 when r is the r -matrix on a quadratic Lie algebra defined by a Lagrangian splitting (see Example 3.2) is proved in [39, Theorem 2.3].

Lemma 3.4. [38] *Let r be a quasitriangular r -matrix on \mathfrak{g} . If $\lambda(r)$ is skew-symmetric, i.e., if $\lambda(r)$ is a bivector field on Y , then it is Poisson, and λ is a left Poisson action of the Lie bialgebra (\mathfrak{g}, δ_r) on the Poisson manifold $(Y, -\lambda(r))$.*

Note that $\lambda(r)$ is skew-symmetric if and only if $\lambda(r + r^{21}) = 0$, or, equivalently,

$$(3.14) \quad (r_+ - r_-)(\mathfrak{q}_y^0) \subset \mathfrak{q}_y, \quad y \in Y,$$

where for $y \in Y$, \mathfrak{q}_y is the stabilizer of λ at y , i.e., $\mathfrak{q}_y = \ker(\lambda_y) \subset \mathfrak{g}$, and \mathfrak{q}_y^0 the annihilator of \mathfrak{q}_y in \mathfrak{g}^* . In particular, (3.14) is independent on the skew-symmetric part of r . Note also that if $r \in \mathfrak{g} \otimes \mathfrak{g}$ is

factorizable defining the symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on \mathfrak{g} , then (3.14) is equivalent to $\mathfrak{q}_y \subset \mathfrak{g}$ being coisotropic with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ for each $y \in Y$, i.e.,

$$(3.15) \quad \mathfrak{q}_y^\perp \subset \mathfrak{q}_y, \quad \forall y \in Y.$$

Definition 3.5. By an *admissible quadruple* we mean a quadruple $(\mathfrak{g}, r, Y, \lambda)$, where \mathfrak{g} is a Lie algebra, $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a quasitriangular r -matrix on \mathfrak{g} , Y is a manifold, and λ is a left Lie algebra action of \mathfrak{g} on Y such that $\lambda(r)$ is skew-symmetric, i.e., (3.14) holds for every $y \in Y$. Given an admissible quadruple $(\mathfrak{g}, r, Y, \lambda)$, we refer to $-\lambda(r)$ (and sometimes $\lambda(r)$) as *the Poisson structure on Y defined by (r, λ)* .

Let G be a connected Lie group with Lie algebra \mathfrak{g} , and let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a quasitriangular r -matrix on \mathfrak{g} . Let r^L (resp. r^R) be the left (resp. right) invariant tensor field on G with value r at the identity element of G . Then the bivector field on G given by

$$\pi_G = r^L - r^R$$

is Poisson, making (G, π_G) into a Poisson Lie group [14]. If $\lambda : G \times Y \rightarrow Y$ is a left Lie group action of G on a manifold Y such that $\lambda(r)$ is skew-symmetric, where $\lambda : \mathfrak{g} \rightarrow \mathcal{V}^1(Y)$ also denotes the induced left Lie algebra action of \mathfrak{g} on Y (see (1.16)), we also call (G, r, Y, λ) an *admissible triple*. In this case, λ is a left Poisson action of the Poisson Lie group (G, π_G) on $(Y, -\lambda(r))$. When the action λ is transitive, we also say that (G, r, Y, λ) is a *homogeneous admissible quadruple*.

Remark 3.6. It is clear from the definition that a quadruple (G, r, Y, λ) is admissible if and only if $(G, r, \mathcal{O}, \lambda)$ is admissible for every G -orbit in \mathcal{O} in Y . In studying admissible quadruples, we may therefore restrict ourselves to homogeneous ones. Let Q be any closed subgroup of G with Lie algebra \mathfrak{q} , and let $\lambda_{G/Q}$ be the left action of G on G/Q by left translation. Then the quadruple $(G, r, G/Q, \lambda_{G/Q})$ is admissible if and only if $(r_+ - r_-)(\mathfrak{q}^0) \subset \mathfrak{q}$, or equivalently, $\mathfrak{q}^\perp \subset \mathfrak{q}$ when r is factorizable. As a special case, assume that the Lie algebra \mathfrak{q} of $Q \subset G$ satisfies $\mathfrak{q} \supset \mathfrak{f}_+$. Then

$$(r_+ - r_-)(\mathfrak{q}^0) \subset (r_+ - r_-)(\mathfrak{f}_+^0) \subset r_+(\mathfrak{f}_+^0) \subset \mathfrak{f}_+ \subset \mathfrak{q},$$

so $(G, r, G/Q, \lambda_{G/Q})$ is admissible. On the other hand, by Lemma 3.1, Q is a Poisson Lie subgroup of (G, π_G) , so π_G projects to a well-defined Poisson structure, denoted by $\pi_{G/Q}$, on G/Q . It is shown in [38, §7] that $\pi_{G/Q} = -\lambda_{G/Q}(r)$. \diamond

Let (G, r, Y, λ) be an admissible quadruple and consider the Poisson structure $\pi = -\lambda(r)$ on Y . By the definition of π and by (3.14), one has

$$(3.16) \quad \pi_y^\#(\alpha_y) = \lambda_y(r_+(\lambda_y^*(\alpha_y))) = \lambda_y(r_-(\lambda_y^*(\alpha_y))), \quad \alpha_y \in T_y^*Y.$$

It follows that for any Lie subalgebra \mathfrak{m} of \mathfrak{g} containing \mathfrak{f}_+ or \mathfrak{f}_- and for any $y \in Y$, one has

$$(3.17) \quad \text{Im}(\pi_y^\#) \subset \lambda_y(\mathfrak{m}) \subset T_yY.$$

The following Lemma 3.7 follows immediately from (3.17) and Lemma 3.4.

Lemma 3.7. *Let (G, r, Y, λ) be an admissible quadruple, and let M be a connected Lie subgroup of G such that the Lie algebra \mathfrak{m} of M contains \mathfrak{f}_+ or \mathfrak{f}_- . Then every orbit \mathcal{O}_M of M in Y is a Poisson submanifold of Y with respect to the Poisson structure $\pi = -\lambda(r)$, and λ restricts to a left Poisson action of the Poisson Lie group $(M, \pi_G|_M)$ on $(\mathcal{O}_M, \pi|_{\mathcal{O}_M})$.*

In particular, by taking $M = G$ in Lemma 3.7, every G -orbit in Y , equipped with the Poisson structure π , is a Poisson homogeneous space of the Poisson Lie group (G, π_G) .

Let \mathcal{O} be any G -orbit in Y . For $y \in \mathcal{O}$, let $(\text{Im}(\pi_y^\#))^0 \subset T_y^* \mathcal{O}$ be the co-normal space in \mathcal{O} at y of the symplectic leaf of π through y . Let $[\lambda_y] : \mathfrak{g}/\mathfrak{q}_y \rightarrow T_y \mathcal{O}$ be the vector space isomorphism induced by $\lambda_y : \mathfrak{g} \rightarrow T_y Y$. Identify $(\mathfrak{g}/\mathfrak{q}_y)^*$ with $\mathfrak{q}_y^0 \subset \mathfrak{g}^*$. Then one has the vector space isomorphism $[\lambda_y]^* : T_y^* \mathcal{O} \rightarrow \mathfrak{q}_y^0$.

Lemma 3.8. *For any $y \in \mathcal{O}$, one has*

$$[\lambda_y]^* ((\text{Im}(\pi_y^\#))^0) = \mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y) = \mathfrak{q}_y^0 \cap r_-^{-1}(\mathfrak{q}_y).$$

Consequently, $\dim \mathcal{O} - \dim(\text{Im}(\pi_y^\#)) = \dim(\mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y)) = \dim(\mathfrak{q}_y^0 \cap r_-^{-1}(\mathfrak{q}_y))$.

Proof. By (3.16), one has $\text{Im}(\pi_y^\#) = \lambda_y(r_+(\mathfrak{q}_y^0)) = [\lambda_y]((\mathfrak{q}_y + r_+(\mathfrak{q}_y^0))/\mathfrak{q}_y)$. Thus

$$[\lambda_y]^* ((\text{Im}(\pi_y^\#))^0) = \mathfrak{q}_y^0 \cap (r_+(\mathfrak{q}_y^0))^0 = \mathfrak{q}_y^0 \cap r_-^{-1}(\mathfrak{q}_y) = \mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y).$$

Q.E.D.

We recall a result of Drinfeld [12] on Poisson homogeneous spaces: suppose that $(\mathcal{O}, \pi, \lambda)$ is a Poisson homogeneous space of a Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$. Then the Lie algebroid $(\mathcal{O} \times \mathfrak{g}) \bowtie T^* \mathcal{O}$ over \mathcal{O} in (2.3) is transitive, so the kernel of its anchor map $-\lambda + \pi^\#$ is a bundle of Lie algebras over \mathcal{O} . Consider the map

$$\Psi : (\mathcal{O} \times \mathfrak{g}) \bowtie T^* \mathcal{O} \longrightarrow \mathfrak{d}, \quad (x, \alpha_y) \longmapsto x + \lambda_y^*(\alpha_y), \quad x \in \mathfrak{g}, y \in \mathcal{O}, \alpha_y \in T_y^* \mathcal{O},$$

where $(\mathfrak{d}, \langle, \rangle_{\mathfrak{d}})$ is the Drinfeld double Lie algebra of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. For $y \in \mathcal{O}$, let

$$\mathfrak{l}_y = \Psi(\ker(-\lambda_y + \pi_y^\#)) = \{x + \xi : x \in \mathfrak{g}, \xi \in \mathfrak{q}_y^0, \lambda_y(x) = \pi_y^\#([\lambda_y]^*)^{-1}(\xi)\} \subset \mathfrak{d}.$$

By [37], for $y \in Y$, $\Psi|_{\ker(-\lambda_y + \pi_y^\#)} : \ker(-\lambda_y + \pi_y^\#) \rightarrow \mathfrak{d}$ is injective and \mathfrak{l}_y is a Lagrangian subalgebra of $(\mathfrak{d}, \langle, \rangle_{\mathfrak{d}})$, called the *Drinfeld Lagrangian subalgebra at y* associated to the Poisson structure π .

The following Lemma 3.9 follows directly from the definition of \mathfrak{l}_y and is basic to Drinfeld's theory on Poisson homogeneous spaces [12].

Lemma 3.9. (i). *The conormal subspace $(\text{Im}(\pi_y^\#))^0 \subset T_y^* \mathcal{O}$ can be identified with $\mathfrak{g}^* \cap \mathfrak{l}_y$ under the vector space isomorphism $[\lambda_y]^* : T_y^* \mathcal{O} \rightarrow \mathfrak{q}_y^0$;*

(ii). *For $x \in \mathfrak{g}$, $\lambda_y(x) \in \text{Im}(\pi_y^\#)$ if and only if $x \in \mathfrak{g} \cap (\mathfrak{g}^* + \mathfrak{l}_y) = \text{pr}_{\mathfrak{g}}(\mathfrak{l}_y)$, where $\text{pr}_{\mathfrak{g}} : \mathfrak{d} \rightarrow \mathfrak{g}$ is the projection with respect to the decomposition $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$.*

In the context of Lemma 3.9, one checks from the definition that

$$(3.18) \quad \mathfrak{l}_y = \{x + \xi \in \mathfrak{d} : x \in \mathfrak{g}, \xi \in \mathfrak{q}_y^0, x + r_+(\xi) \in \mathfrak{q}_y\}.$$

Thus $\mathfrak{g}^* \cap \mathfrak{l}_y = \mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y)$. This gives another proof of Lemma 3.8 using Drinfeld's general theory in [12]. We will return to the Drinfeld Lagrangian subalgebras in §3.4.

3.4. Strongly admissible quadruples.

Definition 3.10. By a *strongly admissible quadruple* we mean a quadruple (G, r, Y, λ) , where G is a connected Lie group, r is a factorizable quasitriangular r -matrix on the Lie algebra \mathfrak{g} of G , Y is a manifold, and λ is a left action of G on Y , such that the stabilizer subgroup Q_y of G at every $y \in Y$ is connected and its Lie algebra \mathfrak{q}_y satisfies

$$(3.19) \quad [\mathfrak{q}_y, \mathfrak{q}_y] \subset \mathfrak{q}_y^\perp \subset \mathfrak{q}.$$

Remark 3.11. A strongly admissible quadruple is thus an admissible quadruple (G, r, Y, λ) with the additional requirements that r be factorizable, the stabilizer subgroup Q_y of G at each $y \in Y$ be connected and its Lie algebra satisfy $[\mathfrak{q}_y, \mathfrak{q}_y] \subset \mathfrak{q}_y^\perp$. \diamond

Example 3.12. Let G be a connected Lie group with Lie algebra \mathfrak{g} and r a factorizable quasitriangular r -matrix on \mathfrak{g} . *Homogeneous* strongly admissible quadruples are then of the form $(G, r, G/Q, \lambda)$, where Q is a closed and connected Lie subgroup of G whose Lie algebra \mathfrak{q} satisfies

$$(3.20) \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}^\perp \subset \mathfrak{q},$$

and $\lambda_{G/Q}$ is the left action of G on G/Q by left translation. In the special case when $r = r_{(u, u')}$ is the r -matrix on \mathfrak{g} defined by a Lagrangian splitting $\mathfrak{g} = \mathfrak{u} + \mathfrak{u}'$ of $(\mathfrak{g}, \langle, \rangle_{\mathfrak{g}})$, Condition (3.20) was first introduced in [39] and some properties of the Poisson structure $\lambda_{G/Q}(r)$ on G/Q were also studied in [39]. Note also that (3.20) holds automatically if \mathfrak{q} is Lagrangian with respect to $\langle, \rangle_{\mathfrak{g}}$. \diamond

Example 3.13. Continuing with Example 3.3, let G be a connected complex simple Lie group with Lie algebra \mathfrak{g} , and let $\langle, \rangle_{\mathfrak{g}}$ be a fixed non-zero scalar multiple of the Killing form of \mathfrak{g} . Recall from Example 3.3 that Lagrangian splittings of $(\mathfrak{g} \oplus \mathfrak{g}, \langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$ of the form $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + \mathfrak{l}$ have been classified by Belavin-Drinfeld in [2]. Let $r_{BD} \in \mathfrak{g} \otimes \mathfrak{g}$ be the factorizable quasitriangular r -matrix on \mathfrak{g} corresponding to such an $\mathfrak{l} \subset \mathfrak{g} \oplus \mathfrak{g}$. Let P be any parabolic subgroup of G . As the Lie algebra \mathfrak{p} of P is coisotropic with respect to $\langle, \rangle_{\mathfrak{g}}$, the quadruple $(G, r_{BD}, G/P, \lambda_{G/P})$ is admissible. On the other hand, the Lie algebra \mathfrak{p} of P satisfies $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}^\perp$ if and only if \mathfrak{p} is Borel. Thus $(G, r_{BD}, G/P, \lambda_{G/P})$ is strongly admissible if and only if P is a Borel subgroup of G . \diamond

We now prove some preliminary properties of strongly admissible quadruples.

Lemma 3.14. *Assume that (G, r, Y, λ) is strongly admissible. Then for any $y_1, y_2 \in Y$ in the same G -orbit in Y , the linear isomorphism*

$$(3.21) \quad I_{y_2, y_1} : \mathfrak{q}_{y_1}/\mathfrak{q}_{y_1}^\perp \longrightarrow \mathfrak{q}_{y_2}/\mathfrak{q}_{y_2}^\perp, \quad x + \mathfrak{q}_{y_1}^\perp \longmapsto \text{Ad}_g(x) + \mathfrak{q}_{y_2}^\perp, \quad x \in \mathfrak{q}_{y_1},$$

is independent of the choice of $g \in G$ such that $gy_1 = y_2$.

Proof. For any $y \in Y$, as the stabilizer subgroup Q_y of G at $y \in Y$ is connected, condition (3.19) implies that the action of Q_y on $\mathfrak{q}_y/\mathfrak{q}_y^\perp$ induced by the adjoint action of Q_y on \mathfrak{q}_y is trivial. Consequently, the map I_{y_2, y_1} is independent of the choice of $g \in G$ such that $gy_1 = y_2$.

Q.E.D.

It is also clear that if y_1, y_2, y_3 are in the same G -orbit, then

$$(3.22) \quad I_{y_2, y_1}^{-1} = I_{y_1, y_2} : \mathfrak{q}_{y_2}/\mathfrak{q}_{y_2}^\perp \longrightarrow \mathfrak{q}_{y_1}/\mathfrak{q}_{y_1}^\perp,$$

$$(3.23) \quad I_{y_3, y_2} \circ I_{y_2, y_1} = I_{y_3, y_1} : \mathfrak{q}_{y_1}/\mathfrak{q}_{y_1}^\perp \longrightarrow \mathfrak{q}_{y_3}/\mathfrak{q}_{y_3}^\perp.$$

Definition 3.15. Let (G, r, Y, λ) be a strongly admissible quadruple. For $y_1, y_2 \in Y$ in the same G -orbit, define

$$(3.24) \quad \mathfrak{l}_{y_1, y_2} = \{(x_1, x_2) \in \mathfrak{q}_{y_1} \oplus \mathfrak{q}_{y_2} : I_{y_2, y_1}(x_1 + \mathfrak{q}_{y_1}^\perp) = x_2 + \mathfrak{q}_{y_2}^\perp\} \subset \mathfrak{g} \oplus \mathfrak{g}.$$

Recall from §3.2 that, as r is factorizable, the Drinfeld double Lie algebra of the Lie bialgebra (\mathfrak{g}, δ_r) can be identified with the quadratic Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}, \langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$, where $\langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}}$ is defined in (3.9).

Lemma 3.16. *Let (G, r, Y, λ) be a strongly admissible quadruple. For any $y_1, y_2 \in Y$ in the same G -orbit, \mathfrak{l}_{y_1, y_2} is a Lagrangian subalgebra of $(\mathfrak{g} \oplus \mathfrak{g}, \langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$. Moreover, for any $y \in Y$,*

$$(3.25) \quad \mathfrak{l}_{y, y} = \{(x_1, x_2) \in \mathfrak{q}_y \oplus \mathfrak{q}_y : x_1 - x_2 \in \mathfrak{q}_y^\perp\}$$

is the Drinfeld Lagrangian subalgebra \mathfrak{l}_y at y associated to the Poisson structure $\pi = -\lambda(r)$ on the G -orbit $Gy \subset Y$.

Proof. Let $y \in Y$ be arbitrary and let $\mathcal{O} = Gy \subset Y$. By (3.18), the Drinfeld Lagrangian subalgebra \mathfrak{l}_y at y associated to the Poisson structure $\pi = -\lambda(r)$ on \mathcal{O} is given by

$$\mathfrak{l}_y = \{(x, x) + (r_+^b(x'), r_-^b(x')) : x \in \mathfrak{g}, x' \in \mathfrak{q}_y^\perp, x + r_+^b(x') \in \mathfrak{q}_y\},$$

from which it follows that $\mathfrak{l}_y = \mathfrak{l}_{y, y}$. For any $y_1, y_2 \in Gy$, let $y_1 = g_1 y$ and $y_2 = g_2 y$, where $g_1, g_2 \in G$. It follows from the definitions that

$$(3.26) \quad \mathfrak{l}_{y_1, y_2} = (\text{Ad}_{g_1} \oplus \text{Ad}_{g_2})(\mathfrak{l}_{y, y}).$$

Thus \mathfrak{l}_{y_1, y_2} is a Lagrangian subalgebra of $(\mathfrak{g} \oplus \mathfrak{g}, \langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$.

Q.E.D.

Remark 3.17. By Lemma 3.9 and Lemma 3.16, for a strongly admissible quadruple (G, r, Y, λ) , the rank of π at $y \in Y$ is equal to $\dim(Gy) - \dim(\mathfrak{l}_r \cap \mathfrak{l}_{y, y})$. \diamond

4. ORBIT INTERSECTIONS FOR STRONGLY ADMISSIBLE QUADRUPLES

4.1. The set-up. Assume that (G, r, Y, λ) is an admissible quadruple, in which r is factorizable. Let M_+ and M_- be connected Lie subgroups of G whose respective Lie algebras \mathfrak{m}_+ and \mathfrak{m}_- satisfy

$$(4.1) \quad \mathfrak{f}_+ \subset \mathfrak{m}_+, \quad \mathfrak{f}_- \subset \mathfrak{m}_-.$$

Let \mathbb{T} be the connected component of $M_+ \cap M_-$ containing the identity element. The Lie algebra \mathfrak{t} of \mathbb{T} is then given by $\mathfrak{t} = \mathfrak{m}_+ \cap \mathfrak{m}_-$. By Lemma 3.1, \mathbb{T} is a Poisson Lie subgroup of (G, π_G) , where $\pi_G = r^L - r^R$, and $(\mathfrak{t}, \delta_r|_{\mathfrak{t}})$ is a sub-Lie bialgebra of the Lie bialgebra (\mathfrak{g}, δ_r) . Consider the decomposition

$$Y = \bigsqcup_{\mathcal{O}_+, \mathcal{O}_-} \mathcal{O}_+ \cap \mathcal{O}_-,$$

where \mathcal{O}_+ and \mathcal{O}_- are respectively M_+ -orbits and M_- -orbits in Y . As r is factorizable, one has $\mathfrak{f}_+ + \mathfrak{f}_- = \mathfrak{g}$, and thus $\mathfrak{m}_+ + \mathfrak{m}_- = \mathfrak{g}$. Consequently, each non-empty intersection $\mathcal{O}_+ \cap \mathcal{O}_-$ is a smooth submanifold of Y , and by Lemma 3.7, also a Poisson submanifold with respect to the Poisson structure $\pi = -\lambda(r)$ on Y . Denote the restriction of π to $\mathcal{O}_+ \cap \mathcal{O}_-$ also by π . As $\mathcal{O}_+ \cap \mathcal{O}_-$ is \mathbb{T} -invariant, the Poisson action λ of (G, π_G) on (Y, π) restricts to a Poisson action on $(\mathcal{O}_+ \cap \mathcal{O}_-, \pi)$ by the Poisson Lie group $(\mathbb{T}, \pi_G|_{\mathbb{T}})$ and the Lie bialgebra $(\mathfrak{t}, \delta_r|_{\mathfrak{t}})$.

Definition 4.1. The six-tuple $(G, r, Y, \lambda, M_+, M_-)$ is said to be *admissible* if the quadruple (G, r, Y, λ) is strongly admissible and if \mathbb{T} is abelian, acting on $(Y, \lambda(r))$ by Poisson isomorphisms, and the \mathbb{T} -leaves of $\lambda(r)$ in Y are precisely all the connected components of non-empty intersections $\mathcal{O}_+ \cap \mathcal{O}_-$ where \mathcal{O}_+ is an M_+ -orbit and \mathcal{O}_- an M_- -orbit in Y .

Remark 4.2. If G is an affine algebraic group over \mathbb{C} and if M_+ and M_- are algebraic subgroups of G such that $M_+ \cap M_-$ is connected, by [45, Corollary 1.5], every non-empty intersection of an M_+ -orbit and an M_- -orbit in Y is irreducible and thus connected. \diamond

Using $\mathfrak{q}_y^\perp \subset \mathfrak{q}_y$, one has

$$\begin{aligned} (\mathfrak{q}_y + \mathfrak{m}_+ \cap (\mathfrak{m}_- + \mathfrak{q}_y^\perp))^\perp &= \mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y) = \mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp \cap \mathfrak{q}_y + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y), \\ (\mathfrak{q}_y + \mathfrak{m}_+ \cap (\mathfrak{m}_- + \mathfrak{q}_y))^\perp &= \mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y^\perp) = \mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp \cap \mathfrak{q}_y^\perp + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y^\perp). \end{aligned}$$

It follows that

$$\delta_y = \dim \left(\frac{\mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp \cap \mathfrak{q}_y + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y)}{\mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp \cap \mathfrak{q}_y^\perp + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y^\perp)} \right).$$

Note that $\mathfrak{m}_+^\perp \cap \mathfrak{m}_-^\perp = (\mathfrak{m}_+ + \mathfrak{m}_-)^\perp = \mathfrak{g}^0 = 0$. Writing an element $x \in \mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp \cap \mathfrak{q}_y + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y)$ uniquely as $x = x_+ + x_-$, where $x_+ \in \mathfrak{m}_+^\perp \cap \mathfrak{q}_y$ and $x_- \in \mathfrak{m}_-^\perp \cap \mathfrak{q}_y$, the map

$$\mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp \cap \mathfrak{q}_y + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y) \longrightarrow \mathfrak{q}_y/\mathfrak{q}_y^\perp, \quad x \longmapsto x_+ + \mathfrak{q}_y^\perp = -x_- + \mathfrak{q}_y^\perp$$

induces a well-defined vector space isomorphism

$$\frac{\mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp \cap \mathfrak{q}_y + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y)}{\mathfrak{q}_y^\perp \cap (\mathfrak{m}_+^\perp \cap \mathfrak{q}_y^\perp + \mathfrak{m}_-^\perp \cap \mathfrak{q}_y^\perp)} \longrightarrow \mathfrak{a}_y^+ \cap \mathfrak{a}_y^-.$$

It follows that $\delta_y = \dim(\mathfrak{a}_y^+ \cap \mathfrak{a}_y^-)$.

Q.E.D.

For the remainder of §4.2, assume that (G, r, Y, λ) is strongly admissible, and let (M_+, M_-) be a pair of connected Lie subgroups of G satisfying (4.1).

Lemma 4.5. *For any M_+ -orbit \mathcal{O}_+ and any M_- -orbit \mathcal{O}_- in Y , one has*

- 1) $I_{y_2, y_1}(\mathfrak{a}_{y_1}^+) = \mathfrak{a}_{y_2}^+$ for all $y_1, y_2 \in \mathcal{O}_+$;
- 2) $I_{y_2, y_1}(\mathfrak{a}_{y_1}^-) = \mathfrak{a}_{y_2}^-$ for all $y_1, y_2 \in \mathcal{O}_-$.

Proof. We only prove 1), the proof of 2) being similar. Assume thus $y_2 = m_+ y_1$, where $m_+ \in M_+$. As $\text{Ad}_{m_+} \mathfrak{m}_+^\perp = \mathfrak{m}_+^\perp$, one has

$$I_{y_2, y_1}(\mathfrak{a}_{y_1}^+) = \frac{\text{Ad}_{m_+}(\mathfrak{m}_+^\perp \cap \mathfrak{q}_{y_1})}{\text{Ad}_{m_+}(\mathfrak{m}_+^\perp \cap \mathfrak{q}_{y_1}^\perp)} = \frac{\mathfrak{m}_+^\perp \cap \mathfrak{q}_{y_2}}{\mathfrak{m}_+^\perp \cap \mathfrak{q}_{y_2}^\perp} = \mathfrak{a}_{y_2}^+.$$

Q.E.D.

Let $(\mathcal{O}_+, \mathcal{O}_-)$ be any pair of M_+ - and M_- -orbits contained in the same G -orbit \mathcal{O} in Y , possibly $\mathcal{O}_+ \cap \mathcal{O}_- = \emptyset$. Let $y_0 \in \mathcal{O}$, $y_+ \in \mathcal{O}_+$ and $y_- \in \mathcal{O}_-$ be arbitrary, and let $g_+, g_- \in G$ be such that $y_+ = g_+ y_0$ and $y_- = g_- y_0$. Define

$$\begin{aligned} (4.6) \quad \delta_{\mathcal{O}_+, \mathcal{O}_-} &= \dim \left((I_{y_0, y_+}(\mathfrak{a}_{y_+}^+)) \cap (I_{y_0, y_-}(\mathfrak{a}_{y_-}^-)) \right) \\ &= \dim \left(\left(\frac{(\text{Ad}_{g_+} \mathfrak{m}_+^\perp) \cap \mathfrak{q}_{y_0}}{(\text{Ad}_{g_+} \mathfrak{m}_+^\perp) \cap \mathfrak{q}_{y_0}^\perp} \right) \cap \left(\frac{(\text{Ad}_{g_-} \mathfrak{m}_-^\perp) \cap \mathfrak{q}_{y_0}}{(\text{Ad}_{g_-} \mathfrak{m}_-^\perp) \cap \mathfrak{q}_{y_0}^\perp} \right) \right), \end{aligned}$$

where the intersection on the right hand side of (4.6) is in the vector space $\mathfrak{q}_{y_0}/\mathfrak{q}_{y_0}^\perp$.

Proposition 4.6. *For any pair $(\mathcal{O}_+, \mathcal{O}_-)$ of M_+ - and M_- -orbits contained in the same G -orbit in Y , the integer $\delta_{\mathcal{O}_+, \mathcal{O}_-}$ in (4.6) is independent of the choices of $y_0 \in \mathcal{O}$, $y_+ \in \mathcal{O}_+$ and $y_- \in \mathcal{O}_-$. Moreover, when $\mathcal{O}_+ \cap \mathcal{O}_- \neq \emptyset$, one has $\delta_{\mathcal{O}_+, \mathcal{O}_-} = \delta_y$ for any $y \in \mathcal{O}_+ \cap \mathcal{O}_-$.*

Let again (M_+, M_-) be a pair of r -admissible Lie subgroups of G with respective Lie algebras \mathfrak{m}_+ and \mathfrak{m}_- . If every M_+ -orbit \mathcal{O}_+ and every M_- -orbit \mathcal{O}_- in G/Q contain elements of the form gQ with $g \in N_G(\mathfrak{c})$, one can use (4.17) to compute the integers $\delta_{\mathcal{O}_+, \mathcal{O}_-}$ and the subspaces $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$ of \mathfrak{t} , as in the following Proposition 4.18.

Proposition 4.18. *Suppose that $\mathfrak{q} = \mathfrak{c} + \mathfrak{q}^\perp$ is a direct decomposition, and assume that all (M_+, Q) - and (M_-, Q) -double cosets in G contain elements in $N_G(\mathfrak{c})$. Let $(\mathcal{O}_+, \mathcal{O}_-)$ be any pair of M_+ and M_- -orbits in G/Q , and choose any $g_+, g_- \in N_G(\mathfrak{c})$ such that $g_+Q \in \mathcal{O}_+$ and $g_-Q \in \mathcal{O}_-$. Then*

$$\begin{aligned}\delta_{\mathcal{O}_+, \mathcal{O}_-} &= \dim \left(p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_{g_+}\mathfrak{q}) \cap \text{Ad}_{g_+g_-^{-1}} \left(p_{\mathfrak{c}}(\mathfrak{m}_-^\perp \cap \text{Ad}_{g_-}\mathfrak{q}) \right) \right), \\ \mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-} &= p_{\mathfrak{t}}(V_{g_+, g_-}),\end{aligned}$$

where $V_{g_+, g_-} = \left\{ (x_+, x_-) \in (\mathfrak{m}_+ \cap \text{Ad}_{g_+}\mathfrak{q}) \oplus (\mathfrak{m}_- \cap \text{Ad}_{g_-}\mathfrak{q}) : p_{\mathfrak{c}}(x_+) = \text{Ad}_{g_+g_-^{-1}}p_{\mathfrak{c}}(x_-) \right\}$. In particular, the six-tuple $(G, r, G/Q, \lambda_{G/Q}, M_+, M_-)$ is admissible if

$$p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_g\mathfrak{q}) \cap \text{Ad}_h \left(p_{\mathfrak{c}}(\mathfrak{m}_-^\perp \cap \text{Ad}_k\mathfrak{q}) \right) = 0, \quad \forall g, h, k \in N_G(\mathfrak{c}).$$

Proof. Proposition 4.18 follows directly from (4.8) for $\delta_{\mathcal{O}_+, \mathcal{O}_-}$ and (4.16) for $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$.

Q.E.D.

Remark 4.19. Note that if \mathfrak{q} is Lagrangian with respect to $\langle, \rangle_{\mathfrak{g}}$, by taking $\mathfrak{c} = 0$, the six-tuple $(G, r, G/Q, \lambda_{G/Q}, M_+, M_-)$ is automatically admissible, and the leaf stabilizer $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$ is given by

$$\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-} = p_{\mathfrak{t}} \left((\mathfrak{m}_+ \cap \text{Ad}_{g_+}\mathfrak{q}) \oplus (\mathfrak{m}_- \cap \text{Ad}_{g_-}\mathfrak{q}) \right)$$

for any $g_+Q \in \mathcal{O}_+$ and $g_-Q \in \mathcal{O}_-$. ◇

Example 4.20. Let (G, π_G) be a connected Poisson Lie group, where $\pi_G = r^L - r^R$ for a factorizable quasitriangular r -matrix r on the Lie algebra \mathfrak{g} of G . Equip the direct product Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ with the direct product factorizable quasitriangular r -matrix $(r, -r)$, and let λ be the left action of $G \times G$ on G given by

$$(g_1, g_2) \cdot g = g_1 g g_2^{-1}, \quad g_1, g_2, g \in G.$$

As the stabilizer subgroup of λ at $g \in G$ is $\{(g_1, g^{-1}g_1) : g_1 \in G\}$, which is connected and its Lie algebra Lagrangian with respect to the symmetric bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$ associated to $(r, -r)$, the quadruple $(G \times G, (r, -r), G, \lambda)$ is strongly admissible. It is easy to see that $-\lambda(r, -r) = \pi_G$. If (M_+, M_-) is a pair of r -admissible Lie subgroups of G , then $(M_+ \times M_+, M_- \times M_-)$ is a pair of $(r, -r)$ -admissible Lie subgroups of $G \times G$. By Remark 4.19, the six-tuple $(G \times G, (r, -r), G, \lambda, M_+ \times M_+, M_- \times M_-)$ is admissible, and hence the (\mathbb{T}, \mathbb{T}) -leaves of (G, π_G) are precisely the connected components of non-empty intersections of (M_+, M_+) -double cosets and (M_-, M_-) -double cosets in G . As a special case, if r is defined by a Lagrangian splitting $\mathfrak{g} = \mathfrak{u} + \mathfrak{u}'$, by taking $M_+ = U'$ and $M_- = U$, where U and U' are the connected Lie subgroups of G with Lie algebras \mathfrak{u} and \mathfrak{u}' respectively, the symplectic leaves of (G, π_G) are precisely the connected components of (U, U) -double cosets and (U', U') -double cosets in G (see, for example, [46]).

5. MIXED PRODUCT POISSON STRUCTURES ASSOCIATED TO ADMISSIBLE QUADRUPLES

5.1. The construction. Let r be a quasitriangular r -matrix on a Lie algebra \mathfrak{g} , and let $n \geq 1$ be any integer. Writing $r = \sum_i x_i \otimes y_i \in \mathfrak{g} \otimes \mathfrak{g}$, we introduced in [38] a quasitriangular r -matrix $r^{(n)}$ on the

direct product Lie algebra $\mathfrak{g}^n = \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ (n -copies), which is given by

$$(5.1) \quad r^{(n)} = \sum_{1 \leq j \leq n, j \text{ is odd}} (r)_j + \sum_{1 \leq j \leq n, j \text{ is even}} (-r^{21})_j - \sum_{1 \leq j < k \leq n} \sum_i (y_i)_j \wedge (x_i)_k,$$

where for $X \in \mathfrak{g}^{\otimes l}$, $l = 1, 2$, and $1 \leq j \leq n$, $(X)_j \in (\mathfrak{g}^n)^{\otimes l}$ is the image of X under the embedding of \mathfrak{g} into \mathfrak{g}^n as the j 'th summand. Since the symmetric part of $r^{(n)}$ is $(s, -s, s, -s, \dots)$, where s is the symmetric part of r , if r is factorizable, so is $r^{(n)}$.

Assume that (G, r, Y_i, λ_i) is an admissible quadruple for each $1 \leq i \leq n$, and consider the product manifold $Y = Y_1 \times \cdots \times Y_n$ and the direct product action λ of \mathfrak{g}^n on Y , i.e.,

$$\lambda : \mathfrak{g}^n \longrightarrow \mathcal{V}^1(Y), \quad \lambda(x_1, x_2, \dots, x_n) = (\lambda_1(x_1), \lambda_2(x_2), \dots, \lambda_n(x_n)), \quad x_j \in \mathfrak{g}.$$

Then the quadruple $(G^n, r^{(n)}, Y, \lambda)$ is admissible. Moreover, when each (G, r, Y_i, λ_i) is strongly admissible, so is $(G^n, r^{(n)}, Y, \lambda)$. Being the sum of the direct product Poisson structure $(-\lambda_1(r), \dots, -\lambda_n(r))$ on Y and some "mixed terms", the Poisson structure $-\lambda(r^{(n)})$ on Y is an example of a *mixed product Poisson structure* on the product manifold Y (see [38]).

In §5, we apply the general theory in §4 to the admissible quadruples $(G^n, r^{(n)}, Y, \lambda)$. We first establish a property of the quasitriangular r -matrix $r^{(n)}$ on \mathfrak{g}^n . For notational simplicity, set

$$(5.2) \quad \tilde{r} = r^{(n)} \in \mathfrak{g}^n \otimes \mathfrak{g}^n, \quad \tilde{\mathfrak{f}}_{\pm} = \text{Im}(\tilde{r}_{\pm}) \subset \mathfrak{g}^n.$$

Recall that when r is factorizable, one has the Lie subalgebra \mathfrak{l}_r of $\mathfrak{g} \oplus \mathfrak{g}$ given by

$$\mathfrak{l}_r = \{(r_+(\xi), r_-(\xi)) : \xi \in \mathfrak{g}^*\} = \{(r_+^b(x), r_-^b(x)) : x \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g},$$

and the Lie subalgebras $\mathfrak{f}_{\pm} = \text{Im}(r_{\pm}^b)$ of \mathfrak{g} . Let τ be the automorphism of \mathfrak{g}^n defined by

$$\tau(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = (x_1, x_3, \dots, x_{n-1}, x_n, x_2), \quad x_j \in \mathfrak{g}.$$

Lemma 5.1. *Assume that r is factorizable. Then*

$$\begin{aligned} \tilde{\mathfrak{f}}_+ &= \tau(\mathfrak{l}_r \oplus \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^{m-1}), & \tilde{\mathfrak{f}}_- &= \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^m, & \text{if } n = 2m \text{ is even,} \\ \tilde{\mathfrak{f}}_+ &= \mathfrak{f}_+ \oplus \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^m, & \tilde{\mathfrak{f}}_- &= \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^m \oplus \mathfrak{f}_-, & \text{if } n = 2m + 1 \text{ is odd.} \end{aligned}$$

Proof. Assume first that $n = 2m$ is even, and, for notational simplicity, set

$$\tilde{\mathfrak{f}}'_+ = \tau(\mathfrak{l}_r \oplus \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^{m-1}) \quad \text{and} \quad \tilde{\mathfrak{f}}'_- = \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^m.$$

Let $\tilde{\xi} = (\xi_1, \xi_2, \dots, \xi_n) \in (\mathfrak{g}^*)^n \cong (\mathfrak{g}^n)^*$, and write

$$(5.3) \quad \tilde{r}_+(\tilde{\xi}) = (x_1, x_2, \dots, x_n), \quad \tilde{r}_-(\tilde{\xi}) = (y_1, y_2, \dots, y_n).$$

By the definition of \tilde{r} , one has, for $1 \leq j \leq n$,

$$(5.4) \quad \begin{aligned} x_j &= \begin{cases} r_-(\xi_1 + \cdots + \xi_{j-1}) + r_+(\xi_j + \xi_{j+1} + \cdots + \xi_n), & j \text{ odd,} \\ r_-(\xi_1 + \cdots + \xi_{j-1} + \xi_j) + r_+(\xi_{j+1} + \cdots + \xi_n), & j \text{ even,} \end{cases} \\ y_j &= \begin{cases} r_-(\xi_1 + \cdots + \xi_{j-1} + \xi_j) + r_+(\xi_{j+1} + \cdots + \xi_n), & j \text{ odd,} \\ r_-(\xi_1 + \cdots + \xi_{j-1}) + r_+(\xi_j + \xi_{j+1} + \cdots + \xi_n), & j \text{ even.} \end{cases} \end{aligned}$$

As $x_1 = r_+(\xi_1 + \cdots + \xi_n)$, $x_n = r_-(\xi_1 + \cdots + \xi_n)$, and for $1 \leq k \leq m-1$,

$$\begin{aligned} x_{2k} &= x_{2k+1} = r_-(\xi_1 + \cdots + \xi_{2k}) + r_+(\xi_{2k+1} + \cdots + \xi_n) \\ &= r_+(\xi_1 + \cdots + \xi_n) + (r_- - r_+)(\xi_1 + \cdots + \xi_{2k}), \end{aligned}$$

one has $\tilde{r}_+(\tilde{\xi}) \in \tilde{\mathfrak{f}}'_+$. Moreover, since $r_- - r_+ : \mathfrak{g}^* \rightarrow \mathfrak{g}$ is invertible, $\tilde{r}_+(\tilde{\xi}) = 0$ if and only if $\xi_{2k-1} + \xi_{2k} = 0$ for every $1 \leq k \leq m$. Thus $\dim \ker \tilde{r}_+ = m(\dim \mathfrak{g})$. It follows that $\dim(\text{Im}(\tilde{r}_+)) = \dim(\tilde{\mathfrak{f}}'_+)$ and hence $\tilde{\mathfrak{f}}_+ = \tilde{\mathfrak{f}}'_+$. Similarly, since

$$y_{2k-1} = y_{2k} = r_-(\xi_1 + \cdots + \xi_{2k-1}) + r_+(\xi_{2k} + \cdots + \xi_n) = (r_- - r_+)(\xi_1 + \cdots + \xi_{2k-1}) + r_+(\xi_1 + \cdots + \xi_n)$$

for every $1 \leq k \leq m$, $\tilde{r}_-(\tilde{\xi}) \in \tilde{\mathfrak{f}}'_-$. Moreover, $\tilde{r}_-(\tilde{\xi}) = 0$ if and only if $y_1 = 0$ and $y_j - y_{j+1} = 0$ for every $1 \leq j \leq n-1$, which is equivalent to $r_-(\xi_1) + r_+(\xi_n) = 0$ and $\xi_{2k} + \xi_{2k+1} = 0$ for every $1 \leq k \leq m-1$. Since the map $(\mathfrak{g}^*)^2 \rightarrow \mathfrak{g}, (\xi, \eta) \mapsto r_-(\xi) + r_+(\eta)$ is surjective, one sees that $\dim(\text{Im}(\tilde{r}_-)) = \dim(\tilde{\mathfrak{f}}'_-)$ and hence $\tilde{\mathfrak{f}}_- = \tilde{\mathfrak{f}}'_-$.

Assume that $n = 2m+1$ is odd, and set $\tilde{\mathfrak{f}}'_+ = \mathfrak{f}_+ \oplus \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^m$ and $\tilde{\mathfrak{f}}'_- = \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^m \oplus \mathfrak{f}_-$. Let $\tilde{\xi} = (\xi_1, \dots, \xi_n) \in (\mathfrak{g}^*)^n$ with $\tilde{r}_+(\tilde{\xi})$ and $\tilde{r}_-(\tilde{\xi})$ as given in (5.3) and (5.4). Again it is clear that $\tilde{r}_+(\tilde{\xi}) \in \tilde{\mathfrak{f}}'_+$. Moreover, $\tilde{r}_+(\tilde{\xi}) = 0$ if and only if $r_+(\xi_n) = 0$ and $\xi_{2k-1} + \xi_{2k} = 0$ for every $1 \leq k \leq m$. It follows by dimension counting that $\tilde{\mathfrak{f}}_+ = \tilde{\mathfrak{f}}'_+$. Similar arguments show that $\tilde{\mathfrak{f}}_- = \tilde{\mathfrak{f}}'_-$.

Q.E.D.

Assume again that r is factorizable, and let (M_+, M_-) be an r -admissible pair of Lie subgroups of G with respective Lie algebras \mathfrak{m}_+ and \mathfrak{m}_- . Let $M_+^{(n)}, M_-^{(n)} \subset G^n$ be given by

$$\begin{aligned} M_+^{(n)} &= M_+ \times \overbrace{G_{\text{diag}} \times \cdots \times G_{\text{diag}}}^{m-1} \times M_-, & M_-^{(n)} &= \overbrace{G_{\text{diag}} \times \cdots \times G_{\text{diag}}}^m, & \text{if } n = 2m \text{ is even,} \\ M_+^{(n)} &= M_+ \times \overbrace{G_{\text{diag}} \times \cdots \times G_{\text{diag}}}^m, & M_-^{(n)} &= \overbrace{G_{\text{diag}} \times \cdots \times G_{\text{diag}}}^m \times M_-, & \text{if } n = 2m+1 \text{ is odd.} \end{aligned}$$

Then their Lie algebras $\mathfrak{m}_+^{(n)}$ and $\mathfrak{m}_-^{(n)}$ are respectively given by

$$\begin{aligned} \mathfrak{m}_+^{(n)} &= \mathfrak{m}_+ \oplus \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^{m-1} \oplus \mathfrak{m}_-, & \mathfrak{m}_-^{(n)} &= \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^m, & \text{if } n = 2m \text{ is even,} \\ \mathfrak{m}_+^{(n)} &= \mathfrak{m}_+ \oplus \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^m, & \mathfrak{m}_-^{(n)} &= \overbrace{\mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}}^m \oplus \mathfrak{m}_-, & \text{if } n = 2m+1 \text{ is odd.} \end{aligned}$$

It is clear from Lemma 5.1 that the pair $(M_+^{(n)}, M_-^{(n)})$ of subgroups of G^n is $r^{(n)}$ -admissible. Let again \mathbb{T} be the connected component of $M_+ \cap M_-$ containing the identity element of G . Then $\mathbb{T}^{(n)} := \{(g, \dots, g) : g \in \mathbb{T}\}$ is the connected component of $M_+^{(n)} \cap M_-^{(n)}$ containing the identity element of G^n .

5.2. A homogeneous case. As in §4.5, consider a six-tuple $(G, r, G/Q, \lambda_{G/Q}, M_+, M_-)$, where G is a connected Lie group with Lie algebra \mathfrak{g} , r a factorizable quasitriangular r -matrix on \mathfrak{g} , Q a closed and connected Lie subgroup of G whose Lie algebra \mathfrak{q} satisfies

$$(5.5) \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}^\perp \subset \mathfrak{q},$$

and (M_+, M_-) a pair of r -admissible Lie subgroups of G with respective Lie algebras \mathfrak{m}_+ and \mathfrak{m}_- . In §5.2, we consider, for each integer $n \geq 1$, the six-tuple

$$(G^n, r^{(n)}, (G/Q)^n, \lambda, M_+^{(n)}, M_-^{(n)}),$$

where λ is the direct product of the action $\lambda_{G/Q}$ of G on each factor of $(G/Q)^n$. Identify

$$\mathbb{T} \xrightarrow{\sim} \mathbb{T}^{(n)}, \quad t \mapsto (t, t, \dots, t), \quad t \in \mathbb{T}.$$

Then \mathbb{T} acts on $(G/Q)^n$ diagonally. As the case of $n = 1$ is covered in Proposition 4.18, we assume that $n \geq 2$.

Assumption 5.2. There exists a direct sum decomposition $\mathfrak{q} = \mathfrak{c} + \mathfrak{q}^\perp$ such that

- 1) every (K, Q) -double cosets in G , where $K \in \{M_+, M_-, Q\}$, contains elements in $N_G(\mathfrak{c})$;
- 2) $p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_g \mathfrak{q}) \cap \text{Ad}_h p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_k \mathfrak{q}) = 0$ for all $g, h, k \in N_G(\mathfrak{c})$,

where recall that $p_{\mathfrak{c}} : \mathfrak{g} \rightarrow \mathfrak{c}$ is the projection with respect to the decomposition $\mathfrak{g} = \mathfrak{c} + \mathfrak{c}^\perp$.

Remark 5.3. Note that by taking $\mathfrak{c} = 0$, Assumption 5.2 holds automatically if $\mathfrak{q} = \mathfrak{q}^\perp$. \diamond

Notation 5.4. For $g = (g_1, \dots, g_n) \in G^n$, let $\underline{g} = (g_1 Q, g_2 Q, \dots, g_n Q) \in (G/Q)^n$. Let $e \in G$ be the identity element G . Let $(\tilde{\mathcal{O}}_+, \tilde{\mathcal{O}}_-)$ be any pair of $M_+^{(n)}$ and $M_-^{(n)}$ -orbits in $(G/Q)^n$. Assumption 5.2 implies that there exist $g, h \in (N_G(\mathfrak{c}))^n$ of the form

$$(5.6) \quad g = \begin{cases} (g_1, e, g_3, e, g_5, \dots, e, g_{2m-1}, g_{2m}), & \text{if } n = 2m \text{ is even} \\ (g_1, e, g_3, e, g_5, \dots, e, g_{2m-1}, e, g_{2m+1}), & \text{if } n = 2m + 1 \text{ is even} \end{cases},$$

$$(5.7) \quad h = \begin{cases} (e, h_2, e, h_4, \dots, e, h_{2m-2}, e, h_{2m}), & \text{if } n = 2m \text{ is even} \\ (e, h_2, e, h_4, \dots, e, h_{2m-2}, e, h_{2m}, h_{2m+1}), & \text{if } n = 2m + 1 \text{ is odd} \end{cases},$$

such that $\underline{g} \in \tilde{\mathcal{O}}_+$ and $\underline{h} \in \tilde{\mathcal{O}}_-$. With $g, h \in (N_G(\mathfrak{c}))^n$ so chosen, let

$$(5.8) \quad g \bowtie h = \begin{cases} (g_1, h_2, g_3, h_4, \dots, g_{2m-1}, h_{2m}), & \text{if } n = 2m \text{ is even} \\ (g_1, h_2, g_3, h_4, \dots, g_{2m-1}, h_{2m}, g_{2m+1}), & \text{if } n = 2m + 1 \text{ is odd} \end{cases},$$

and let $(g \bowtie h)_{n+1} = g_n$ if n is even and $(g \bowtie h)_{n+1} = h_n$ if n is odd. Let $c = (c_1, c_2, \dots, c_{n+1})$, where $(c_1, c_2, \dots, c_n) = g \bowtie h$ and $c_{n+1} = (g \bowtie h)_{n+1}$, and let

$$(5.9) \quad V_c = \left\{ (x_+, x_-) \in (\mathfrak{m}_+ \cap \text{Ad}_{c_1} \mathfrak{q}) \oplus (\mathfrak{m}_- \cap \text{Ad}_{c_{n+1}} \mathfrak{q}) : p_{\mathfrak{c}}(x_+) = \text{Ad}_{c_1 c_2 \dots c_n c_{n+1}^{-1}}(p_{\mathfrak{c}}(x_-)) \right\}.$$

Recall again that $\mathfrak{t} = \mathfrak{m}_+ \cap \mathfrak{m}_-$, and one has the projection $p_{\mathfrak{t}} : \mathfrak{m}_+ \oplus \mathfrak{m}_- \rightarrow \mathfrak{t} \cong \mathfrak{t}_{\text{diag}}$ with respect to the decomposition $\mathfrak{m}_- \oplus \mathfrak{m}_- = \mathfrak{t}_{\text{diag}} + \mathfrak{l}_r$.

Proposition 5.5. *Under Assumption 5.2,*

- (1) the six-tuple $(G^n, r^{(n)}, (G/Q)^n, \lambda, M_+^{(n)}, M_-^{(n)})$ is admissible for every $n \geq 2$;
- (2) for any pair $(\tilde{\mathcal{O}}_+, \tilde{\mathcal{O}}_-)$ of $M_+^{(n)}$ and $M_-^{(n)}$ -orbits in $(G/Q)^n$, the leaf stabilizer of every \mathbb{T} -leaf in $\tilde{\mathcal{O}}_+ \cap \tilde{\mathcal{O}}_-$ is given by $\mathfrak{t}_{\tilde{\mathcal{O}}_+, \tilde{\mathcal{O}}_-} = p_{\mathfrak{t}}(V_c) \subset \mathfrak{t}$ with $V_c \subset \mathfrak{m}_+ \oplus \mathfrak{m}_-$ given in (5.9).

Proof. Let $(\tilde{\mathcal{O}}_+, \tilde{\mathcal{O}}_-)$ be an arbitrary pair of $M_+^{(n)}$ and $M_-^{(n)}$ -orbits in $(G/Q)^n$, and let $g, h \in (N_G(\mathfrak{c}))^n$ be as in (5.6) and (5.7) such that $y_+ = \underline{g} \in \tilde{\mathcal{O}}_+$ and $y_- = \underline{h} \in \tilde{\mathcal{O}}_-$. We use the description in Lemma 4.17 of the Lagrangian subalgebra \mathfrak{l}_{y_+, y_-} of $\mathfrak{g}^n \oplus \mathfrak{g}^n$ to show that $\delta_{\tilde{\mathcal{O}}_+, \tilde{\mathcal{O}}_-} = 0$ and to compute the subspace $\mathfrak{t}_{\tilde{\mathcal{O}}_+, \tilde{\mathcal{O}}_-}$ of $\mathfrak{t} = \mathfrak{m}_+ \cap \mathfrak{m}_-$. For notational simplicity, set $\tilde{M}_\pm := M_\pm^{(n)}$ and let $\tilde{\mathfrak{m}}_\pm$ be their respective Lie algebras.

Assume first that $n = 2m$ is even. By Lemma 4.17, $(\tilde{\mathfrak{m}}_+^\perp \oplus \tilde{\mathfrak{m}}_-^\perp) \cap \mathfrak{l}_{y_+, y_-}$ consists of elements $a \in \mathfrak{g}^n \oplus \mathfrak{g}^n$ of the form

$$(5.10) \quad a = (x_1 + x'_1, x_2 + x'_2, \dots, x_n + x'_n, z_1 + z'_1, z_2 + z'_2, \dots, z_n + z'_n),$$

where, by writing $g = (g_1, g_2, \dots, g_n)$ and $h = (h_1, h_2, \dots, h_n)$,

$$x_j, z_j \in \mathfrak{c}, \quad x'_j \in \text{Ad}_{g_j} \mathfrak{q}^\perp, \quad z'_j \in \text{Ad}_{h_j} \mathfrak{q}^\perp, \quad x_j = \text{Ad}_{g_j h_j^{-1}} z_j, \quad j = 1, \dots, n,$$

$x_1 + x'_1 \in \mathfrak{m}_+^\perp \cap \text{Ad}_{g_1} \mathfrak{q}$, $x_{2m} + x'_{2m} \in \mathfrak{m}_+^\perp \cap \text{Ad}_{g_{2m}} \mathfrak{q}$, and

$$x_{2j} = x_{2j+1}, \quad x'_{2j} = x'_{2j+1}, \quad j = 1, \dots, m-1,$$

$$z_{2j-1} = z_{2j}, \quad z'_{2j-1} = z'_{2j}, \quad j = 1, \dots, m.$$

It follows that

$$x_1 = \text{Ad}_{g_1 h_2 g_3 h_4 \dots g_{2m-1} h_{2m} g_{2m}^{-1}}(x_{2m}) \in p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_{c_1} \mathfrak{q}) \cap \text{Ad}_{c_1 c_2 \dots c_n c_{n+1}^{-1}} p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_{c_{n+1}} \mathfrak{q}).$$

By 2) of Assumption 5.2, $x_1 = 0$, and it follows that $x_j = 0$ for every $j = 1, \dots, n$. By (4.8), $\delta_{\tilde{\mathfrak{o}}_+, \tilde{\mathfrak{o}}_-} = 0$. The case of $n = 2m + 1$ is odd is proved similarly. This proves (1).

To prove (2), note that by Lemma 5.1,

$$\mathfrak{t}_{\tilde{\mathfrak{o}}_+, \tilde{\mathfrak{o}}_-} = p_{\mathfrak{t}}((p(\tilde{\mathfrak{m}}_+ \oplus \tilde{\mathfrak{m}}_-)) \cap \mathfrak{l}_{y_+, y_-}),$$

where $p : \mathfrak{g}^n \oplus \mathfrak{g}^n \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ is given by

$$p(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n) = \begin{cases} (a_1, a_n), & \text{if } n \text{ is even,} \\ (a_1, b_n), & \text{if } n \text{ is odd} \end{cases}.$$

Replacing $\tilde{\mathfrak{m}}_+^\perp \oplus \tilde{\mathfrak{m}}_-^\perp$ in the proof of (1) by $\tilde{\mathfrak{m}}_+ \oplus \tilde{\mathfrak{m}}_-$, one sees that

$$p((\tilde{\mathfrak{m}}_+ \oplus \tilde{\mathfrak{m}}_-) \cap \mathfrak{l}_{y_+, y_-}) \subset V_c.$$

Assume again that $n = 2m$ is even and let $(x_+, x_-) \in V_c$. Then there is a unique element $a \in (\tilde{\mathfrak{m}}_+ \oplus \tilde{\mathfrak{m}}_-) \cap \mathfrak{l}_{y_+, y_-}$ of the form (5.10) with $x_1 + x'_1 = x_+$, $x_n + x'_n = x_-$, $x'_j = 0$ for $j = 2, \dots, n-1$, and $z'_j = 0$ for all $j = 1, \dots, n$. Moreover, $p(a) = (x_+, x_-)$. This shows that $p((\tilde{\mathfrak{m}}_+ \oplus \tilde{\mathfrak{m}}_-) \cap \mathfrak{l}_{y_+, y_-}) = V_c$ when n is even, and by Proposition 4.15, one has $\mathfrak{t}_{\tilde{\mathfrak{o}}_+, \tilde{\mathfrak{o}}_-} = p_{\mathfrak{t}}(V_c) \subset \mathfrak{t}$. The case when n is odd is proved similarly. This proves (2).

Q.E.D.

5.3. Main examples. Consider again a homogeneous strongly admissible quadruple $(G, r, G/Q, \lambda_{G/Q})$, but where we assume that the Lie algebra \mathfrak{q} of Q satisfies

$$(5.11) \quad \mathfrak{f}_+ \subset \mathfrak{q} \quad \text{and} \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}^\perp.$$

Since $\mathfrak{q}^\perp \subset \mathfrak{f}_+^\perp \subset \mathfrak{f}_+ \subset \mathfrak{q}$, the quadruple $(G, r, G/Q, \lambda_{G/Q})$ is indeed strongly admissible. Moreover, by Remark 3.6, Q is a Poisson Lie subgroup of the Poisson Lie group (G, π_G) , where $\pi_G = r^L - r^R$, and the Poisson structure $-\lambda_{G/Q}(r)$ on G/Q coincides with the projection $\pi_{G/Q}$ of π_G to G/Q .

Let $n \geq 1$ be an integer, and let G^n acts on itself from the right by (1.2). Then we have the two quotient manifolds

$$Y_n = G \times_Q \dots \times_Q G/Q \quad \text{and} \quad X_n = G \times_Q \dots \times_Q G$$

of G^n , each with the quotient Poisson structure, respectively denoted by π_{Y_n} and π_{X_n} , which, by definition, are the projections of the direct product Poisson structure π_G^n on G^n . Let (M_+, M_-) be a pair of r -admissible Lie subgroups of G , and let again \mathbb{T} be the connected component of $M_+ \cap M_-$ containing the identity element. Then \mathbb{T} acts on (Y_n, π_{Y_n}) and $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ acts on (X_n, π_{X_n}) by Poisson isomorphisms via

$$\begin{aligned} t \cdot [g_1, g_2, \dots, g_n]_{Y_n} &= [tg_1, g_2, \dots, g_n]_{Y_n}, \quad t \in \mathbb{T}, g_j \in G, \\ (t_1, t_2) \cdot [g_1, g_2, \dots, g_n]_{X_n} &= [t_1 g_1, g_2, \dots, g_n t_2^{-1}]_{X_n}, \quad t_1, t_2 \in \mathbb{T}, g_j \in G. \end{aligned}$$

In this section, we study the \mathbb{T} -leaves of (Y_n, π_{Y_n}) and the \mathbb{T}^2 -leaves of (X_n, π_{X_n}) . As the case for (Y_1, π_{Y_1}) is covered in Proposition 4.18 and the case of $(X_1 = G, \pi_{X_1} = \pi_G)$ is covered in Example 4.20, we will assume that $n \geq 2$.

We first look at the case of (Y_n, π_{Y_n}) . Consider the diffeomorphism

$$(5.12) \quad J_{Y_n} : Y_n \longrightarrow (G/Q)^n, \quad [g_1, g_2, \dots, g_n]_{Y_n} \longmapsto (g_1 Q, g_1 g_2 Q, \dots, g_1 g_2 \dots g_n Q).$$

Let $\lambda : \mathfrak{g}^n \rightarrow \mathcal{V}^1((G/Q)^n)$ be again the direct product of the action $\lambda_{G/Q}$ on each factor. We have the following crucial Proposition 5.6 from [38, §8].

Proposition 5.6. *As Poisson structures on $(G/Q)^n$, one has $J_{Y_n}(\pi_{Y_n}) = -\lambda(r^{(n)})$.*

We can thus apply Proposition 5.5 to the Poisson structure $\lambda(r^{(n)})$ on $(G/Q)^n$ and use

$$J_{Y_n}^{-1} : (G/Q)^n \longrightarrow Y_n, \quad (k_1Q, k_2Q, \dots, k_nQ) \longmapsto [k_1, k_1^{-1}k_2, k_2^{-1}k_3, \dots, k_{n-1}^{-1}k_n]_{Y_n},$$

to translate the results to the Poisson structure π_{Y_n} on Y_n . For $a = (a_1, a_2, \dots, a_n) \in G^n$, let

$$C(a) = (M_+a_1Q) \times_Q (Qa_2Q) \times_Q \cdots \times_Q (Qa_nQ)/Q \subset Y_n.$$

Let $\mu_{Y_n} : Y_n \rightarrow G/Q$ be given by $\mu_{Y_n}([a_1, a_2, \dots, a_n]_{Y_n}) = a_1a_2 \cdots a_nQ$.

Lemma 5.7. *For any $g, h \in G^n$ of the form (5.6) and (5.7), one has*

$$J_{Y_n}^{-1} \left(\left(M_+^{(n)} \underline{g} \right) \cap \left(M_-^{(n)} \underline{h} \right) \right) = C(g \bowtie h) \cap \mu_{Y_n}^{-1}(M_-(g \bowtie h)_{n+1}Q/Q).$$

Proof. Write $g = (g_1, g_2, \dots, g_n)$ and $h = (h_1, h_2, \dots, h_n)$, and consider first the case when $n = 2m$ is even. Let $k = (k_1, k_2, \dots, k_n) \in G^n$. Then

1) $\underline{k} \in M_+^{(n)} \underline{g}$ if and only if $k_1 \in M_+g_1Q$ and $k_{2i}^{-1}k_{2i+1} \in Qg_{2i}^{-1}g_{2i+1}Q$ for $i = 1, \dots, m-1$, and $k_{2m} \in M_-g_{2m}Q$, and

2) $\underline{k} \in M_-^{(n)} \underline{g}$ if and only if $k_{2j-1}^{-1}k_{2j} \in Qg_{2j-1}^{-1}g_{2j}Q$ for $j = 1, \dots, m$.

It follows that $\underline{k} \in \left(M_+^{(n)} \underline{g} \right) \cap \left(M_-^{(n)} \underline{h} \right)$ if and only if $J_{Y_n}(\underline{k}) \in C(g \bowtie h) \cap \mu_{Y_n}^{-1}(M_-g_nQ/Q)$. The case when $n = 2m+1$ is odd is proved similarly.

Q.E.D.

Theorem 5.8. *Under Assumption 5.2, the following holds for every $n \geq 2$:*

(a) *Every non-empty intersection*

$$(5.13) \quad Y_n(c) = ((M_+c_1Q) \times_Q (Qc_2Q) \times_Q \cdots \times_Q (Qc_nQ)/Q) \cap \mu_{Y_n}^{-1}(M_-c_{n+1}Q/Q) \subset Y_n,$$

where $c = (c_1, \dots, c_{n+1}) \in (N_G(\mathfrak{c}))^{n+1}$, is transversal, and their connected components are precisely all the \mathbb{T} -leaves of π_{Y_n} in Y_n ;

(b) *The leaf stabilizer for every \mathbb{T} -leaf in $Y_n(c)$ in (5.13) is $p_{\mathfrak{c}}(V_c)$, where*

$$V_c = \{(x_+, x_-) \in (\mathfrak{m}_+ \cap \text{Ad}_{c_1}\mathfrak{q}) \oplus (\mathfrak{m}_- \cap \text{Ad}_{c_{n+1}}\mathfrak{q}) : p_{\mathfrak{c}}(x_+) = \text{Ad}_{c_1c_2 \cdots c_n c_{n+1}^{-1}}(p_{\mathfrak{c}}(x_-))\}.$$

Moreover, if G is an affine algebraic group over \mathbb{C} and M_+, M_- and Q are algebraic subgroups such that $M_+ \cap M_-$ is connected, then every non-empty intersection $Y_n(c)$ in (5.13) is irreducible and is thus a single \mathbb{T} -leaf of π_{Y_n} in Y_n .

Proof. Parts (a) and (b) follow immediately from Proposition 5.5 and Lemma 5.7 by relabeling $g \bowtie h$ by (c_1, \dots, c_n) and $(g \bowtie h)_{n+1}$ by c_{n+1} . The last part of Theorem 5.8 follows from Remark 4.2.

Q.E.D.

We now turn to the case of (X_n, π_{X_n}) , where $n \geq 2$. Let

$$\mu_{X_n} : X_n \longrightarrow G, \quad [g_1, g_2, \dots, g_n]_{X_n} \longmapsto g_1g_2 \cdots g_n, \quad g_j \in G.$$

Assumption 5.9. There exists a direct sum decomposition $\mathfrak{q} = \mathfrak{c} + \mathfrak{q}^\perp$ such that

1) every (K, Q) -double cosets in G , where $K \in \{M_+, Q\}$, contains elements in $N_G(\mathfrak{c})$;

2) $p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_g\mathfrak{q}) \cap \text{Ad}_h p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_k\mathfrak{q}) = 0$ for all $g, h, k \in N_G(\mathfrak{c})$,

where again $p_{\mathfrak{c}} : \mathfrak{g} \rightarrow \mathfrak{c}$ is the projection with respect to the decomposition $\mathfrak{g} = \mathfrak{c} + \mathfrak{c}^\perp$.

Theorem 5.10. *Under Assumption 5.9, the following holds for every $n \geq 2$:*

(a) *Every non-empty intersection*

$$(5.14) \quad X_n(c) = ((M_+ c_1 Q) \times_Q (Q c_2 Q) \times_Q \cdots \times_Q (Q c_n M_+)) \cap \mu_{X_n}^{-1}(M_- c_{n+1} M_-) \subset X_n,$$

where $c = (c_1, \dots, c_{n+1})$ with $(c_1, \dots, c_n) \in (N_G(\mathfrak{c}))^n$ and $c_{n+1} \in G$, is transversal, and their connected components are precisely all the \mathbb{T}^2 -leaves of π_{X_n} in X_n ;

(b) *The leaf stabilizer for every \mathbb{T}^2 -leaf in $X_n(c)$ in (5.14) is $(p_t \oplus p_t)(V_c)$, where*

$$V_c = \left\{ \left(x_+, x_-, z_+, \text{Ad}_{c_{n+1}^{-1}}(x_-) \right) : \begin{aligned} &x_+ \in \mathfrak{m}_+ \cap \text{Ad}_{c_1} \mathfrak{q}, \quad x_- \in \mathfrak{m}_- \cap \text{Ad}_{c_{n+1}} \mathfrak{m}_-, \\ &z_+ \in \mathfrak{m}_+ \cap \text{Ad}_{c_n} \mathfrak{q}, \quad p_c(x_+) = \text{Ad}_{c_1 c_2 \cdots c_n}(p_c(z_+)) \end{aligned} \right\}.$$

Moreover, if G is an affine algebraic group over \mathbb{C} and M_+, M_- and Q are algebraic subgroups such that $M_+ \cap M_-$ is connected, then every non-empty intersection $X_n(c)$ in (5.14) is irreducible and is thus a single \mathbb{T}^2 -leaf of π_{X_n} in X_n .

Proof. Consider the diffeomorphism $J_{X_n} : X_n \rightarrow (G/Q)^{n-1} \times G$ given by

$$J_{X_n}([g_1, g_2, \dots, g_n]_{X_n}) = (g_1 Q, g_1 g_2 Q, \dots, g_1 g_2 \cdots g_{n-1} Q, g_1 g_2 \cdots g_n), \quad g_j \in G.$$

We will again study π_{X_n} by studying the Poisson structure $J_{X_n}(\pi_{X_n})$ on $(G/Q)^{n-1} \times G$. Let λ be the left action of G^{n+1} on the product manifold $(G/Q)^{n-1} \times G$ by

$$(g_1, \dots, g_n, g_{n+1}) \cdot (h_1 Q, \dots, h_{n-1} Q, h_n) = (g_1 h_1 Q, \dots, g_{n-1} h_{n-1} Q, g_n h_n g_{n+1}^{-1}),$$

where $g_j, h_k \in G$, and define the direct sum quasitriangular r -matrix $r^{(n+1)}$ on \mathfrak{g}^{n+1} by

$$(5.15) \quad r^{(n+1)} = \begin{cases} (r^{(n)}, 0) + (0, -r), & \text{if } n = 2m + 1 \text{ is odd,} \\ (r^{(n)}, 0) + (0, r^{21}), & \text{if } n = 2m \text{ is even.} \end{cases}$$

Then the homogeneous quadruple $(G^{n+1}, r^{(n+1)}, (G/Q)^{n-1} \times G, \lambda)$ is strongly admissible. It is proved in [38, §8] that, as Poisson structures on $(G/Q)^{n-1} \times G$,

$$(5.16) \quad J_{X_n}(\pi_{X_n}) = -\lambda(r^{(n+1)}).$$

Define the Lie subgroups $M_+^{(n+1)} \subset G^{n+1}$ and $M_-^{(n+1)} \subset G^{n+1}$ respectively by

$$\begin{aligned} M_+^{(n+1)} &= M_+^{(n)} \times M_- & \text{and} & \quad M_-^{(n+1)} = M_-^{(n)} \times M_+ & \text{if } n = 2m \text{ is even,} \\ M_+^{(n+1)} &= M_+^{(n)} \times M_+ & \text{and} & \quad M_-^{(n+1)} = M_-^{(n)} \times M_- & \text{if } n = 2m + 1 \text{ is odd.} \end{aligned}$$

Then the pair $(M_+^{(n+1)}, M_-^{(n+1)})$ is $r^{(n+1)}$ -admissible, and one thus has the six-tuple

$$(5.17) \quad (G^{n+1}, r^{(n+1)}, (G/Q)^{n-1} \times G, \lambda, M_+^{(n+1)}, M_-^{(n+1)}).$$

Same as for Theorem 5.8, we will apply Theorem 1.5 to the six-tuple in (5.17) and use the diffeomorphism $J_{X_n} : X_n \rightarrow (G/Q)^{n-1} \times G$ to prove Theorem 5.10. Note that by identifying

$$\mathbb{T}^2 \xrightarrow{\sim} M_+^{(n+1)} \cap M_-^{(n+1)}, \quad (t_1, t_2) \mapsto (t_1, t_1, \dots, t_1, t_2), \quad t_1, t_2 \in \mathbb{T}$$

the diffeomorphism $J_{X_n} : X_n \rightarrow (G/Q)^{n-1} \times G$ is \mathbb{T}^2 -equivariant. For notational simplicity, set again $\widetilde{M}_\pm = M_\pm^{(n+1)}$ and let $\widetilde{\mathfrak{m}}_\pm$ be their respective Lie algebras.

Assume first that $n = 2m$ is even with $m \geq 1$. Let $(\widetilde{\mathcal{O}}_+, \widetilde{\mathcal{O}}_-)$ be an arbitrary pair of \widetilde{M}_+ and \widetilde{M}_- -orbits in $(G/Q)^{n-1} \times G$. By 1) of Assumption 5.9, there exist

$$\begin{aligned} g &= (g_1, e, g_3, e, g_5, \dots, e, g_{2m-1}, g_{2m}, e) \in (N_G(\mathfrak{c}))^{n-1} \times G \times G, \\ h &= (e, h_2, e, h_4, e, \dots, h_{2m-2}, e, h_{2m}, e) \in (N_G(\mathfrak{c}))^{n+1}, \end{aligned}$$

such that $y_+ := \underline{g} \in \tilde{\mathcal{O}}_+$ and $y_- := \underline{h} \in \tilde{\mathcal{O}}_-$, where $\underline{a} = (a_1 Q, \dots, a_{n-1} Q, a_n a_{n+1}^{-1}) \in (G/Q)^{n-1} \times G$ for $a = (a_1, \dots, a_{n+1}) \in G^{n+1}$. Let

$$c = (g_1, h_2, g_3, h_4, \dots, g_{2m-1}, h_{2m}, g_{2m}) \in (N_G(\mathfrak{c}))^n \times G.$$

By Lemma 4.17, $(\tilde{\mathfrak{m}}_+^\perp \oplus \tilde{\mathfrak{m}}_-^\perp) \cap \mathfrak{l}_{y_+, y_-}$ consists of elements $a \in \mathfrak{g}^{n+1} \oplus \mathfrak{g}^{n+1}$ of the form

$$(5.18) \quad a = (x_1 + x'_1, \dots, x_{n-1} + x'_{n-1}, \text{Ad}_{g_n}(x_n), x_n, z_1 + z'_1, \dots, z_{n-1} + z'_{n-1}, \text{Ad}_{h_n}(z_n), z_n),$$

where, by writing $g = (g_1, g_2, \dots, g_{n+1})$ and $h = (h_1, h_2, \dots, h_{n+1})$,

$$x_j, z_j \in \mathfrak{c}, \quad x'_j \in \text{Ad}_{g_j} \mathfrak{q}^\perp, \quad z'_j \in \text{Ad}_{h_j} \mathfrak{q}^\perp, \quad x_j = \text{Ad}_{g_j h_j^{-1}} z_j, \quad j = 1, \dots, n-1,$$

$x_1 + x'_1 \in \mathfrak{m}_+^\perp \cap \text{Ad}_{g_1} \mathfrak{q}$, $x_n \in \mathfrak{m}_+^\perp \cap \text{Ad}_{g_n^{-1}} \mathfrak{m}_+^\perp$, $z_n \in \mathfrak{m}_+^\perp$, $z_{n-1} + z'_{n-1} = \text{Ad}_{h_n}(z_n)$, and

$$x_{2j} = x_{2j+1}, \quad x'_{2j} = x'_{2j+1}, \quad z_{2j-1} = z_{2j}, \quad z'_{2j-1} = z'_{2j}, \quad j = 1, \dots, m-1.$$

It follows that

$$x_1 = \text{Ad}_{g_1 h_2 g_3 h_4 \dots g_{2m-1} h_{2m}}(z_{2m}) \in p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_{c_1} \mathfrak{q}) \cap \text{Ad}_{c_1 c_2 \dots c_n} p_{\mathfrak{c}}(\mathfrak{m}_+^\perp \cap \text{Ad}_{c_n^{-1}} \mathfrak{q}).$$

By 2) of Assumption 5.9, $x_1 = 0$, and it follows that $x_j = 0$ for every $j = 1, \dots, n-1$. By (4.8), $\delta_{\tilde{\mathcal{O}}_+, \tilde{\mathcal{O}}_-} = 0$. Let $p_{\mathfrak{t} \oplus \mathfrak{t}}$ be the projection from $\tilde{\mathfrak{m}}_+ \oplus \tilde{\mathfrak{m}}_-$ to $\tilde{\mathfrak{m}}_+ \cap \tilde{\mathfrak{m}}_- \cong \mathfrak{t} \oplus \mathfrak{t}$. An argument similar to that in the proof of Theorem 5.8 also shows that

$$p_{\mathfrak{t} \oplus \mathfrak{t}}((\tilde{\mathfrak{m}}_+ \oplus \tilde{\mathfrak{m}}_-) \cap \mathfrak{l}_{y_+, y_-}) = (p_{\mathfrak{t}} \oplus p_{\mathfrak{t}})(V_c).$$

Moreover, an argument similar to that in the proof of Lemma 5.7 shows that $J_{X_n}^{-1}(\tilde{\mathcal{O}}_+ \cap \tilde{\mathcal{O}}_-) = X_n(c)$. Theorem 5.10 is now a consequence of Theorem 1.5. The case of $n = 2m + 1$ is odd is proved similarly.

Q.E.D.

6. APPLICATIONS TO POISSON STRUCTURES RELATED TO FLAG VARIETIES

After reviewing the standard complex semi-simple Lie groups, we prove Theorem 1.1 - Theorem 1.4 stated in §1.2 on the four series of Poisson manifolds given in (1.1).

6.1. Standard complex semi-simple Poisson Lie groups. Let \mathfrak{g} be a complex semi-simple Lie algebra, and let $\langle, \rangle_{\mathfrak{g}}$ be a fixed symmetric non-degenerate invariant bilinear form on \mathfrak{g} . Fix also a choice $(\mathfrak{b}, \mathfrak{b}_-)$ of opposite Borel subalgebras of \mathfrak{g} and let $\mathfrak{h} = \mathfrak{b} \cap \mathfrak{b}_-$, a Cartan subalgebra of \mathfrak{g} . Let Δ and $\Delta_+ \subset \Delta$ be respectively the set of roots for the pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{b}, \mathfrak{h})$, and let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha + \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$ be the corresponding root decomposition. Let $\mathfrak{n}_+ = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$. Equip again the direct product Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ with the bilinear form $\langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}}$ as in (3.9). Then

$$\mathfrak{l}_{\text{st}} = \{(x_+ + x_0, -x_0 + x_-) : x_{\pm} \in \mathfrak{n}_{\pm}, x_0 \in \mathfrak{h}\}$$

is a Lagrangian subalgebra of $(\mathfrak{g} \oplus \mathfrak{g}, \langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$ such that $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + \mathfrak{l}_{\text{st}}$. The decomposition $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + \mathfrak{l}_{\text{st}}$ is called a *standard* Lagrangian splitting of $(\mathfrak{g} \oplus \mathfrak{g}, \langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$ and the element $r_{\text{st}} \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\mathfrak{l}_{r_{\text{st}}} = \mathfrak{l}_{\text{st}}$ a *standard (factorizable quasitriangular) r-matrix* on \mathfrak{g} . Let $\{h_i\}_{i=1}^r$ be a basis of \mathfrak{h} such that $\langle h_i, h_j \rangle_{\mathfrak{g}} = \delta_{ij}$ for $1 \leq i, j \leq r$, and let $E_\alpha \in \mathfrak{g}_\alpha$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be root vectors for $\alpha \in \Delta_+$ such that $\langle E_\alpha, E_{-\alpha} \rangle_{\mathfrak{g}} = 1$. Then r_{st} is explicitly given by

$$(6.1) \quad r_{\text{st}} = \frac{1}{2} \sum_{i=1}^r h_i \otimes h_i + \sum_{\alpha \in \Delta_+} E_{-\alpha} \otimes E_\alpha.$$

It is clear from the definition in (3.4) that the Lie subalgebras \mathfrak{f}_- and \mathfrak{f}_+ of \mathfrak{g} associated to r_{st} are respectively given by $\mathfrak{f}_- = \mathfrak{b}_-$ and $\mathfrak{f}_+ = \mathfrak{b}_+$. It is also clear that r_{st} is factorizable and that the symmetric bilinear form on \mathfrak{g} associated to r_{st} (see §3.2) is precisely $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

By (5.1), one also has the factorizable quasitriangular r -matrix $r_{\text{st}}^{(2)}$ on $\mathfrak{g} \oplus \mathfrak{g}$. Explicitly,

$$(6.2) \quad \begin{aligned} r_{\text{st}}^{(2)} &= \frac{1}{2} \sum_i ((h_i, 0) \otimes (h_i, 0) - (0, h_i) \otimes (0, h_i) - (h_i, 0) \wedge (0, h_i)) \\ &\quad + \sum_{\alpha \in \Delta_+} ((E_{-\alpha} \otimes E_{\alpha}, 0) - (0, E_{\alpha} \otimes E_{-\alpha}) - (E_{\alpha}, 0) \wedge (0, E_{-\alpha})) \\ &= \frac{1}{2} \sum_i (h_i, h_i) \otimes (h_i, -h_i) + \sum_{\alpha \in \Delta_+} ((E_{\alpha}, E_{\alpha}) \otimes (0, -E_{-\alpha}) + (E_{-\alpha}, E_{-\alpha}) \otimes (E_{\alpha}, 0)). \end{aligned}$$

One checks directly (see also [38, §6]) that $r_{\text{st}}^{(2)}$ coincides with the r -matrix on $\mathfrak{g} \oplus \mathfrak{g}$ defined by the Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + \mathfrak{l}_{\text{st}}$ (see the definition in Example 3.2). The Lie subalgebras \mathfrak{f}_+ and \mathfrak{f}_- of $\mathfrak{g} \oplus \mathfrak{g}$ associated to $r_{\text{st}}^{(2)}$ are then \mathfrak{l}_{st} and $\mathfrak{g}_{\text{diag}}$, respectively.

Let G be any connected complex Lie group with Lie algebra \mathfrak{g} . The Poisson structure

$$\pi_{\text{st}} := r_{\text{st}}^L - r_{\text{st}}^R$$

on G is called a *standard multiplicative holomorphic Poisson structure* on G , and the pair (G, π_{st}) a *standard complex semi-simple Poisson Lie group*. One also has the *Drinfeld double* Poisson Lie group $(G \times G, \Pi_{\text{st}})$ of the Poisson Lie group (G, π_{st}) , where

$$\Pi_{\text{st}} = \left(r_{\text{st}}^{(2)} \right)^L - \left(r_{\text{st}}^{(2)} \right)^R.$$

Let B and B_- be the Borel subgroups of G with Lie algebras \mathfrak{b} and \mathfrak{b}_- respectively, and let $T = B \cap B_-$, a maximal torus of G . Let $Q = B$ for the Poisson Lie group (G, π_{st}) and $Q = B \times B_-$ for the Poisson Lie group $(G \times G, \Pi_{\text{st}})$. Then Condition (5.11) is satisfied, and one arrives at the four quotient Poisson manifolds (F_n, π_n) , (\mathbb{F}_n, Π_n) , $(\tilde{F}_n, \tilde{\pi}_n)$ and $(\tilde{\mathbb{F}}_n, \tilde{\Pi}_n)$, as in (1.1) in §1.2.

For (F_n, π_n) and $(\tilde{F}_n, \tilde{\pi}_n)$, take $M_+ = B$ and $M_- = B_-$. Then $M_+ \cap M_- = T$. The projection $p_t : \mathfrak{m}_+ \oplus \mathfrak{m}_- \rightarrow \mathfrak{t} = \mathfrak{h}$ in (4.14) is now given by

$$(6.3) \quad p_t : \mathfrak{b} \oplus \mathfrak{b}_- \longrightarrow \mathfrak{h}, \quad (x_0 + x_+, y_0 + y_-) \longmapsto x_0 + y_0, \quad x_0, y_0 \in \mathfrak{h}, x_+ \in \mathfrak{n}, y_- \in \mathfrak{n}_-.$$

For (\mathbb{F}_n, Π_n) and $(\tilde{\mathbb{F}}_n, \tilde{\Pi}_n)$, take $M_+ = B \times B_-$ and $M_- = G_{\text{diag}} = \{(g, g) : g \in G\}$. Then $M_+ \cap M_- = T_{\text{diag}} = \{(t, t) : t \in T\}$. The projection $p_t : \mathfrak{m}_+ \oplus \mathfrak{m}_- \rightarrow \mathfrak{t} = \mathfrak{h}_{\text{diag}} \cong \mathfrak{h}$ in (4.14) in this case given by

$$(6.4) \quad p_t : (\mathfrak{b} \oplus \mathfrak{b}_-) \oplus \mathfrak{g}_{\text{diag}} \longrightarrow \mathfrak{h}, \quad ((x_0 + x_+, y_0 + y_-), (x, x)) \longmapsto x_0 + y_0,$$

where $x_0, y_0 \in \mathfrak{h}$, $x_+ \in \mathfrak{n}$, $y_- \in \mathfrak{n}_-$, and $x \in \mathfrak{g}$.

We make some further preparation for the proofs of Theorem 1.1 - Theorem 1.4.

Lemma 6.1. *For $u_1, \dots, u_n, v_1, \dots, v_n, w \in W$, the following are equivalent:*

- 1) $(Bu_1Bu_2 \cdots Bu_nB) \cap (B_-v_1B_-v_2 \cdots B_-v_nB_-wB) \neq \emptyset$;
- 2) $w \leq (v_1 * \cdots * v_n)^{-1} * u_1 * \cdots * u_n$.

Proof. For $n = 1$, the equivalent between 1) and 2) is proved in [48, Proposition 4.1]. Assume that $n \geq 2$ and write $u = u_1 * \cdots * u_n$ and $v = v_1 * \cdots * v_n$. Suppose that 1) holds. As

$$Bu_1B \cdots Bu_nB = \bigsqcup_{x \in \mathcal{U}} BxB \quad \text{and} \quad B_-v_1B_-v_2 \cdots B_-v_nB_- = \bigsqcup_{y \in \mathcal{V}} B_-yB_-,$$

where $\mathcal{U} \subset \{x \in W : x \leq u\}$ and $\mathcal{V} \subset \{y \in W : y \leq v\}$, there exist $x \leq u$ and $y \leq v$ such that $(BxB) \cap (B_yB_wB) \neq \emptyset$, and hence $w \leq y^{-1} * x \leq v^{-1} * u$. Conversely, if $w \leq v^{-1} * u$, then $(BuB) \cap (B_vB_wB) \neq \emptyset$. As $BuB \subset Bu_1B \cdots Bu_nB$ and $B_vB \subset B_{v_1}B \cdots B_{v_n}B$, 1) holds.

Q.E.D.

Lemma 6.2. *For any $u, v \in W$ and any conjugacy class C in G , $(BuBB_vB_-) \cap C \neq \emptyset$.*

Proof. As $B_- \cap (BuB) \neq \emptyset$, one has $B_- \cap (BuBB_-) \neq \emptyset$, which implies that $BB_- \subset BuBB_-$. Similarly, $BB_- \subset BB_vB_-$. Thus $B \subset BB_- \subset BuBB_- \subset BuBB_vB_-$. As $B \cap C \neq \emptyset$, one has $(BuBB_vB_-) \cap C \neq \emptyset$.

Q.E.D.

For $n \geq 1$, consider $P_n : (G \times G)^n \rightarrow G^n$ given by $P_n(g_1, k_1, \dots, g_n, k_n) = (g_1, \dots, g_n)$, and the induced projections, both denoted as $[P_n]$, from \mathbb{F}_n to F_n and from $\tilde{\mathbb{F}}_n$ to \tilde{F}_n , i.e.,

$$(6.5) \quad P_n([g_1, k_1, \dots, g_n, k_n]_{\mathbb{F}_n}) = [g_1, \dots, g_n]_{F_n}, \quad P_n([g_1, k_1, \dots, g_n, k_n]_{\tilde{\mathbb{F}}_n}) = [g_1, \dots, g_n]_{\tilde{F}_n}.$$

As $P_1 : (G \times G, \Pi_{\text{st}}) \rightarrow (G, \pi_{\text{st}})$ is Poisson (this follows, for example, from the fact that $P_1(r_{\text{st}}^{(2)}) = r_{\text{st}}$), one knows that $[P_n] : (\mathbb{F}_n, \Pi_n) \rightarrow (F_n, \pi_n)$ and $[P_n] : (\tilde{\mathbb{F}}_n, \tilde{\Pi}_n) \rightarrow (\tilde{F}_n, \tilde{\pi}_n)$ are Poisson. This observation will be used to show that Theorem 1.1 is a special case of Theorem 1.2 (see §6.2) and that Theorem 1.3 a special case of Theorem 1.4 (see §6.3).

6.2. Proofs of Theorem 1.1 and Theorem 1.2. We first prove Theorem 1.2 by applying Theorem 5.8 to the Poisson Lie group $(G \times G, \Pi_{\text{st}})$ and by taking $Q = M_+ = B \times B_-$ and $M_- = G_{\text{diag}}$. As $M_+ \cap M_- = T_{\text{diag}} := \{(t, t) : t \in \mathbb{T}\}$, the intersection $R_w^{\mathbf{u}, \mathbf{v}}$, whenever non-empty, is smooth and connected and has dimension $l(\mathbf{u}) + l(\mathbf{v}) - l(w)$. The fact that $R_w^{\mathbf{u}, \mathbf{v}} \neq \emptyset$ if and only if $w \leq (v_1 * \cdots * v_n)^{-1} * u_1 * \cdots * u_n$ follows directly from Lemma 6.1 (it is also proved in [42, Proposition 3.32] using distinguished double subexpressions). This proves 1) of Theorem 1.2. Letting $c = (u_1, v_1, \dots, u_n, v_n)$ and $c_{n+1} = (w, e)$ in (b) of Theorem 5.8, one proves the first part of 2) of Theorem 1.2. Computing explicitly the subspace $V_c \subset \mathfrak{m}_+ \oplus \mathfrak{m}_-$ as described in Theorem 5.8 and using (6.4) for $p_t : \mathfrak{m}_+ \oplus \mathfrak{m}_- \rightarrow \mathfrak{t} = \mathfrak{h}$, one sees that the leaf stabilizer of $\lambda_{\mathbb{F}_n}$ in $R_w^{\mathbf{u}, \mathbf{v}}$ is precisely $\mathfrak{h}_w^{\mathbf{u}, \mathbf{v}} \subset \mathfrak{h}$. This proves 3) of Theorem 1.2. The second part of 2) follows from 3) by Theorem 5.8. This finishes the proof of Theorem 1.2.

Theorem 1.3 is proved either similarly, by applying Theorem 5.8 to the Poisson Lie group (G, π_{st}) and taking $Q = M_+ = B$ and $M_- = B_-$, or by the following observation: let

$$\hat{F}_n := (G \times B_-) \times_{(B \times B_-)} \cdots \times_{(B \times B_-)} (G \times B_-) / (B \times B_-) \subset \mathbb{F}_n.$$

By Lemma 3.1, $G \times B_-$ is a Poisson Lie subgroup of $(G \times G, \Pi_{\text{st}})$. Thus \hat{F}_n is a Poisson submanifold of (\mathbb{F}_n, Π_n) . It is then clear that the Poisson morphism $[P_n] : (\mathbb{F}_n, \Pi_n) \rightarrow (F_n, \pi_n)$ in (6.5) restricts to a Poisson isomorphism from (\hat{F}_n, Π_n) to (F_n, π_n) . Theorem 1.1 now follows by applying Theorem 1.2 to the Poisson submanifold (\hat{F}_n, Π_n) of (\mathbb{F}_n, Π_n) .

6.3. Proofs of Theorem 1.3 and Theorem 1.4. We first prove Theorem 1.4 by applying Theorem 5.10 to the Poisson Lie group $(G \times G, \Pi_{\text{st}})$ and taking $Q = M_+ = B \times B_-$ and $M_- = G_{\text{diag}}$. As $M_+ \cap M_- = T_{\text{diag}}$, the intersection $R_C^{\mathbf{u}, \mathbf{v}}$, whenever non-empty, is smooth and connected, and

$$\dim R_C^{\mathbf{u}, \mathbf{v}} = \dim((B \times B_-)(\mathbf{u}, \mathbf{v})(B \times B_-)) + \dim \mu_{\tilde{\mathbb{F}}_n}^{-1}(\Omega_C) - \dim \tilde{\mathbb{F}}_n = l(\mathbf{u}) + l(\mathbf{v}) + \dim C + \dim T.$$

Let $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n) \in W^n$, and $C \in \mathcal{C}$ be arbitrary. By definition,

$$\mu_{\tilde{\mathbb{F}}_n}^{-1}(\Omega_C) = \{[g_1, k_2, \dots, g_n, k_n]_{\tilde{\mathbb{F}}_n} : (g_1 g_1 \cdots g_n)(k_1 k_2 \cdots k_n)^{-1} \in C\}.$$

By Lemma 6.2, $(Bu_1Bu_2 \cdots Bu_nBB_-v_n^{-1} \cdots B_-v_2^{-1}B_-v_1^{-1}B_-) \cap C \neq \emptyset$, so $R_C^{\mathbf{u}, \mathbf{v}} \neq \emptyset$. This proves 1) of Theorem 1.4. Letting $c = (u_1, v_1, \dots, u_n, v_n)$ and $c_{n+1} = (c, e)$ in (b) of Theorem 5.10, where $c \in C$ is arbitrary, one sees that the $R_C^{\mathbf{u}, \mathbf{v}}$'s are precisely the T^2 -leaves of $\tilde{\Pi}_n$ in $\tilde{\mathbb{F}}_n$, where T^2 -acts on $\tilde{\mathbb{F}}_n$ by

$$(6.6) \quad (t_1, t_2) \cdot [g_1, k_1, \dots, g_n, k_n]_{\tilde{\mathbb{F}}_n} = [tg_1, tk_1, g_2, k_2, \dots, g_{n-1}, k_{n-1}, g_nt_2^{-1}, k_nt_2^{-1}]_{\tilde{\mathbb{F}}_n}.$$

Computing explicitly the subspace $V_c \subset \mathfrak{m}_+ \oplus \mathfrak{m}_-$ as described in Theorem 5.10 and using (6.4) for $p_t : \mathfrak{m}_+ \oplus \mathfrak{m}_- \rightarrow \mathfrak{t} = \mathfrak{h}$, one sees that the leaf stabilizer for the T^2 -action on $R_C^{\mathbf{u}, \mathbf{v}}$ is given by

$$(\mathfrak{h} \oplus \mathfrak{h})^{\mathbf{u}, \mathbf{v}} := \{(u(x) + v(y), x + y) : x, y \in \mathfrak{h}\},$$

where $u = u_1 * \cdots * u_n$ and $v = v_1 * \cdots * v_n$. Note that $(\mathfrak{h} \oplus 0) + (\mathfrak{h} \oplus \mathfrak{h})^{\mathbf{u}, \mathbf{v}} = \mathfrak{h} \oplus \mathfrak{h}$, the action $\lambda_{\tilde{\mathbb{F}}_n}$ of $T \cong T \times \{e\} \subset T \times T$ on $\tilde{\mathbb{F}}_n$ given in (1.9) is also full, so each $R_C^{\mathbf{u}, \mathbf{v}}$ is a single T -leaf for the action $\lambda_{\tilde{\mathbb{F}}_n}$. Moreover, the leaf-stabilizer of $\lambda_{\tilde{\mathbb{F}}_n}$ is $R_C^{\mathbf{u}, \mathbf{v}}$ is

$$(\mathfrak{h} \oplus 0) \cap ((\mathfrak{h} \oplus \mathfrak{h})^{\mathbf{u}, \mathbf{v}}) = \{u(x) - v(x) : x \in \mathfrak{h}\} = \mathfrak{h}^{\mathbf{u}, \mathbf{v}}.$$

This finishes the proof of Theorem 1.4.

Theorem 1.3 is proved either similarly, by applying Theorem 5.10 to the Poisson Lie group (G, π_{st}) and taking $Q = M_+ = B$ and $M_- = B_-$, or by the following observation: let

$$F_n^\vee := ((G \times B_-) \times_{(B \times B_-)} \cdots \times_{(B \times B_-)} (G \times B_-) \times_{(B \times B_-)} (G \times G)) \cap \mu_{\tilde{\mathbb{F}}_n}^{-1}(G_{\text{diag}}) \subset \tilde{\mathbb{F}}_n.$$

By Theorem 1.4, F_n^\vee is a Poisson submanifold of $(\tilde{\mathbb{F}}_n, \tilde{\Pi}_n)$. It is clear that the Poisson morphism $[P_n] : (\tilde{\mathbb{F}}_n, \tilde{\Pi}_n) \rightarrow (\tilde{F}_n, \tilde{\pi}_n)$ in (6.5) restricts to a Poisson isomorphism from $(F_n^\vee, \tilde{\Pi}_n)$ to $(\tilde{F}_n, \tilde{\pi}_n)$. Theorem 1.3 now follows by applying Theorem 1.4 to the Poisson submanifold $(F_n^\vee, \tilde{\Pi}_n)$ of $(\tilde{\mathbb{F}}_n, \tilde{\Pi}_n)$.

6.4. Other examples. The results in §4 and §5 can be used to give a unified approach to many other examples, old or new, of T -Poisson manifolds related to real or complex semi-simple Lie groups. We give two such example here, leaving other examples to be treated elsewhere.

Example 6.3. Let (G, π_{st}) be again a standard complex semi-simple Lie group as in §6.1, and let $\theta \in \text{Aut}(G)$ be such that $\theta(T) = T$ and $\theta(B) = B$, and denote by the same letter the induced automorphism on \mathfrak{g} . Let λ be the left action of $G \times G$ on G given by

$$(6.7) \quad (g_1, g_2) \cdot_\theta g = g_1 g \theta(g_2)^{-1}, \quad g_1, g_2, g \in G.$$

Orbits of $G_{\text{diag}} \subset G \times G$ on G under λ are called θ -twisted conjugacy classes of G . Let $r = r_{\text{st}}^{(2)}$ be the r -matrix on $\mathfrak{g} \oplus \mathfrak{g}$ defined by the Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + \mathfrak{l}_{\text{st}}$ (see (6.2)). As the stabilizer subalgebras for λ are Lagrangian with respect to $\langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}}$, the quadruple $(G \times G, r, G, \lambda)$ is strongly admissible. The Poisson structure $\pi_\theta = -\lambda(r)$ is studied in [36].

Recall that the Lie subalgebras \mathfrak{f}_- and \mathfrak{f}_+ of $\mathfrak{g} \oplus \mathfrak{g}$ associated to r is $\mathfrak{f}_- = \mathfrak{g}_{\text{diag}}$ and $\mathfrak{f}_+ = \mathfrak{l}_{\text{st}}$. Let $M_- = G_{\text{diag}}$ and let $M_+ = B \times B_-$, so that $M_- \cap M_+ = T_{\text{diag}} = \{(t, t) : t \in T\}$ which we identify with T . Note that M_- -orbits in G are precisely the θ -twisted conjugacy classes in G , and each M_+ -orbit is of the form BwB_- for a unique $w \in W$. It is shown in [7, §2.4] that for a θ -twisted conjugacy class C , there is a unique element $m_C \in W$ such that for $w \in W$, $C \cap (BwB_-) \neq \emptyset$ if and only if $w \leq m_C$. Again as the stabilizer subalgebras of λ are Lagrangian with respect to $\langle, \rangle_{\mathfrak{g} \oplus \mathfrak{g}}$, it follows trivially from Proposition 4.6 that $\delta_{\mathcal{O}_+, \mathcal{O}_-} = 0$ for every $\mathcal{O}_- = C$ and $\mathcal{O}_+ = BwB_-$. Thus the T -leaf decomposition of (G, π_θ) is given by $G = \bigsqcup_{C, w} C \cap (BwB_-)$, where C is a θ -twisted conjugacy class in G and $w \in W$ such that $w \leq m_C$. Denote by $\mathfrak{t}_{C, w}$ the leaf stabilizer of $C \cap (BwB_-)$ in \mathfrak{h} . Pick any $g \in C$ and let $y_- = g$ and $y_+ = \dot{w}$. Then

$$\mathfrak{l}_{y_+, y_-} = \mathfrak{q}_{y_+} \oplus \mathfrak{q}_{y_-} = \text{Ad}_{(\dot{w}, e)} \mathfrak{g}_\theta \oplus \text{Ad}_{(g, e)} \mathfrak{g}_\theta \subset (\mathfrak{g} \oplus \mathfrak{g}) \oplus (\mathfrak{g} \oplus \mathfrak{g}),$$

where $\mathfrak{g}_\theta = \{(\theta(x), x) : x \in \mathfrak{g}\}$. It follows from (4.16) that $\mathfrak{t}_{C,w} = \text{Im}(w\theta + 1)$, a result obtained in [36].

Example 6.4. Let G be a connected complex semi-simple Lie group with Lie algebra \mathfrak{g} , and let $\text{Im}\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be the imaginary part of Killing form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ of \mathfrak{g} . Let $G = KAN$ be an Iwasawa decomposition of G , and let $\mathfrak{k}, \mathfrak{a}$, and \mathfrak{n} be respectively the Lie algebras of K , A , and N . Regarding $(\mathfrak{g}, \text{Im}\langle \cdot, \cdot \rangle_{\mathfrak{g}})$ as a real quadratic Lie algebra, one has the Lagrangian splitting $\mathfrak{g} = \mathfrak{k} + (\mathfrak{a} + \mathfrak{n})$ of $(\mathfrak{g}, \text{Im}\langle \cdot, \cdot \rangle_{\mathfrak{g}})$. Let π_G be the real analytic Poisson structure on G given by $\pi_G = (r_0)^L - (r_0)^R$, where $r_0 \in \mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g}$ is the quasitriangular r -matrix on \mathfrak{g} defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{k} + (\mathfrak{a} + \mathfrak{n})$. Then K is a Poisson Lie subgroup of (G, π_G) , and for each integer $n \geq 1$, one has the quotient space $\mathcal{P}_n = G \times_K \cdots \times_K G/K$ with the quotient Poisson structure $\pi_{\mathcal{P}_n}$ on \mathcal{P}_n . One can regard \mathcal{P}_n as the space of n -gons in the Riemannian symmetric space G/K (see [1, 27]). Taking $M_+ = K$ and $M_- = AN$ so that $M_+ \cap M_- = \{e\}$, it follows from Theorem 5.8 and the decomposition $G = KAK$ that the symplectic leaves of $\pi_{\mathcal{P}_n}$ in \mathcal{P}_n are of the forms $(Ka_1K) \times_K \cdots \times_K (Ka_nK)/K$ with $a = (a_1, \dots, a_n) \in A^n$, the space of n -gons with fixed side lengths a .

REFERENCES

- [1] A. Alekseev, E. Meinrenken, and C. Woodward, Linearization of Poisson actions and singular values of matrix products, *Ann. Inst. Fourier*, Grenoble, **51** (6) (2001), 1691 - 1717.
- [2] A. Belavin and V. Drinfeld, Triangle equations and simple Lie algebras, *Soviet Sci. Rev. Sect. C: Math. Phys. Rev.* **4** (1984) 93 - 165.
- [3] A. Berenstein, S. Fomin, and A. Zelevinsky, Cluster algebras III, Upper bounds and double Bruhat cells, *Duke Math. J.* **126** (2005), 1 - 52.
- [4] S. Billey and I. Coskun, Singularities of generalized Richardson varieties, *Comm. Algebra*, **40** (2012), 1466 - 1495.
- [5] M. Brion and V. Lakshmibai, A geometric approach to standard monomial theory, *Repr. Theory* **7** (2003), 651 - 680.
- [6] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics CRM Barcelona, Basel (2002), Birkhauser.
- [7] K. Y. Chan, J.-H. Lu, and S. To, On intersections of conjugacy classes and Bruhat cells, *Trans. Groups* **15** (2) (2010), 243 - 260.
- [8] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University press, 1994.
- [9] C. De Concini, V. Kac, and C. Procesi, Some quantum analogs of solvable Lie groups, *Geometry and Analysis (Bombay, 1992)*, *Tata. Inst. Fund. Res., Bombay*, (1995), 41 - 65.
- [10] V. G. Drinfeld, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations, *Soviet Math. Dokl.* **27** (1) (1982), 68 - 71.
- [11] V. G. Drinfeld, Quantum groups, *Proceedings of the International Congress of Mathematicians*, **1** (2) (Berkeley, Calif., 1986), 798 - 820, Amer. Math. Soc., Providence, RI, 1987.
- [12] V. G. Drinfeld, On Poisson homogeneous spaces of Poisson-Lie groups, *Theo. Math. Phys.* **95** (2) (1993), 226 - 227.
- [13] B. Elek and J.-H. Lu, On a Poisson structure on Bott-Samelson varieties: computations in coordinates, arXiv:1601.00047.
- [14] P. Etingof and O. Schiffmann, *Lectures on quantum groups*, 2nd edition, international press, 2002.
- [15] S. Evens and J.-H. Lu, On the variety of Lagrangian subalgebras, II, *Ann. Sci. École Norm. Sup.* **39** (2) (2006), 347 - 379.
- [16] S. Evens and J.-H. Lu, Poisson geometry of the Grothendieck resolution of a complex semi-simple group, *Moscow Math. J.* **7** (4) (special volume in honor of V. Ginzburg's 50'th birthday), 613 - 642 (2007).
- [17] R. Fernandes, Lie Algebroids, Holonomy and Characteristic Classes, *Adv. Math.* **170** (1) (2002), 119 - 179.
- [18] S. Fomin and A. Zelevinsky, Double Bruhat Cells and total positivity, *J. Amer. Math. Soc.* **12** (1999), 335 - 380.
- [19] K. Goodearl and E. Letzler, The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras, *Trans. Amer. Math. Soc.*, **352** (2000), 1381 - 1403.
- [20] K. Goodearl and M. Yakimov, Poisson structures on affine spaces and Flag varieties, II, the general case, *Trans. Amer. Math. Soc.* **361** (11) (2009) 5753 - 5780.
- [21] K. Goodearl and M. Yakimov, Quantum cluster algebras and quantum nilpotent algebras, *Proc. Natl. Acad. Sci. USA* **111** (27) (2014), 9696 - 9703.
- [22] N. Hitchin, Instantons, Poisson structures, and generalized Kahler geometry, *Comm. Math. Phys.* **256** (2006), 131 - 164.
- [23] T. J. Hodges and T. Levasseur, Primitive ideals of $C_q[SL(3)]$, *Comm. Math. Phys.* **156** (3) (1993), 581 - 605.
- [24] T. J. Hodges, T. Levasseur, and M. Toro, Algebra structure of multiparameter quantum groups, *Adv. math.* **126** (1997), 52 - 92.

- [25] T. Hoffmann, J. Kellendonk, N. Kutz, and N. Reshetikhin, Factorization dynamics and Coxeter-Toda lattices, *Comm. Math. Phys.* **212** (2) (2000), 297 - 321.
- [26] A. Joseph, On the prime and primitive spectra of the algebra of functions on a quantum group, *J. Alg.* **169** (1994), 441 - 511.
- [27] M. Kapovich, B. Leeb, and J. Millson, The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra, *Memoirs of AMS* **192** (2008).
- [28] C. Kim, *Deformations of Compact Holomorphic Poisson Manifolds and Algebraic Poisson Schemes*, PhD thesis, University of California, Riverside, 2014.
- [29] A. Knutson, T. Lam, and D. Speyer, Projections of Richardson varieties, *J. Reine Angew. Math.* **2014** (687), 133 - 157.
- [30] M. Kogan and A. Zelevinsky, On symplectic leaves and integrable systems in standard complex semi-simple Poisson-Lie groups, *Int. Math. Res. Notices* **2002** (32), 1685 - 1702.
- [31] V. Kreimann and V. Lakshmibai, Richardson varieties in the Grassmannian, *Contributions to Automorphic Forms, Geometry and Number Theory: Shalika-fest, 2002*, Johns Hopkins University Press (2003), 573 - 597.
- [32] V. Lakshmibai and P. Littelman, Richardson varieties and equivariant K -theory, *J. Algebra* **260** (2003), 230 - 260.
- [33] T. Lenagan and M. Yakimov, Prime factors of quantum Schubert cell algebras and clusters for quantum Richardson varieties. arXiv:1503.06297.
- [34] D. Li-Bland and P. Severa, Quasi-Hamiltonian groupoids and multiplicative Manin pairs, *Int. Math. Res. Notices* **2011** (20), 2295 - 2350.
- [35] J.-H. Lu, On a Dimension Formula for Twisted Spherical Conjugacy Classes in Semisimple Algebraic Groups, *Math. Z.* **269** (3-4) (2011), 1181 - 1188.
- [36] J.-H. Lu, On the T -leaves and the ranks of a Poisson structure on twisted conjugacy classes, *Indagationes Math.* **25** (5) (2014), 1102 - 1121.
- [37] J.-H. Lu, Poisson homogeneous spaces and Lie algebroids associated to Poisson actions, *Duke Math. J.* **86** (2) (1997), 261 - 304.
- [38] J.-H. Lu and V. Mouquin, Mixed product Poisson structures associated to Poisson Lie groups and Lie bialgebras, arXiv:1504.06843 [math.DG].
- [39] J.-H. Lu and M. Yakimov, Group orbits and regular partitions of Poisson manifolds, *Comm. Math. Phys.* **283** (3) (2008), 729 - 748.
- [40] K. Mackenzie, Lie algebroids and Lie pseudoalgebras, *Bull. London Math. Soc.* **27** (1995), 97 - 147.
- [41] A. Mériaux and G. Cauchon, Admissible diagrams in $U_q^w(\mathfrak{g})$ and combinatoric properties of Weyl groups, *Represent. Theo.* **14** (2010), 645 - 687.
- [42] V. Mouquin, Cell decompositions of double Bott-Samelson varieties, *Int. Math. Res. Notices* **2015** (18), 8372 - 8410.
- [43] V. Mouquin, *On a Deodhar-type decomposition and a Poisson structure on double Bott-Samelson varieties*, PhD thesis of the University of Hong Kong, 2013.
- [44] N. Reshetikhin and M. Semenov-Tian-Shansky, Quantum R -matrices and factorization problems, *J. Geom. Phys.* **5** (1988), 533 - 550.
- [45] R. Richardson, Intersections of double cosets in algebraic groups, *Indag. Math. (N.S.)* **3** (1) (1992), 69 - 77.
- [46] M. Semenov-Tian-Shansky, Dressing transformation and Poisson group actions, *Publ. Res. Inst. Math. Sci.* **21** (6) (1985), 1237 - 1260.
- [47] M. Semenov-Tian-Shansky, *Integrable systems: the r -matrix approach*, RIMS, Kyoto University (2008).
- [48] B. Webster and M. Yakimov, A Deodhar-type stratification on the double flag variety, *Trans. Groups* **2007** (4), 769 - 785.
- [49] M. Yakimov, A classification of H -primes of quantum partial flag varieties, *Proc. Amer. Math. Soc.* **138** (2010), 1249 - 1261.
- [50] M. Yakimov, Invariant prime ideals in quantizations of nilpotent Lie algebras, *Proc. London Math. Soc. (3)* **101** (2) (2010), 454 - 476.
- [51] M. Yakimov, Strata of prime ideals of De Concini-Kac-Procesi algebras and Poisson geometry, *New trends in noncommutative algebras*, eds: P. Ara et al., Contemp. Math. **562** (2012), 265 - 278.
- [52] M. Yakimov, Weak splittings of quotients of Drinfeld and Heisenberg doubles, in *Developments and retrospectives in Lie theory, geometric and analytic methods*, Eds: G. Mason et al, (2014), 245 - 268.

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